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SYMMETRY OF PHYSICAL LAWS: PART 2, Q-NUMBER THEORY OF SPACE-TIME INVERSIONS AND CHARGE-CONJUGATION.

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SYMMETRY OF PHYSICAL LAWS

PART II

Q-Number Theory of Space-Time Inversions and Charge-Conjugation

by

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Symmetry of Physical Laws

Part II

- Q-Number Theory of Space-Time Inversions
and
Charge Conjugation -

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Abstract

Symmetry properties of physical laws with respect to space-inversion, time-reversal and charge-conjugation are investigated in detail in the framework of the quantized field theory. In most cases, the requirements of invariance for these transformations are automatically satisfied, but in a few other cases, they can impose certain conditions on the ways in which different types of interaction are to be mixed. The ideas of space-parity, charge-parity and superselection rules are coherently derived from the general formulation of field theory. #1. Introduction. #2. Mathematical preliminaries. #3. Operators for reversion, mirage, inversion and charge-conjugation. #4. Passage from c-number theory to q-number theory. #5. Pion field. #6. Electromagnetic field. #7. Charged Spinor field -— Linear momentum representation. #8. Charged spinor field — Angular momentum representation. #9. Tensorial quantities built with spinors. #10. Nucleon-pion

interaction. #11. Nucleon-lepton interaction. #12. Selection rules and superselection rules. #13. Illustrations --- Disintegration of positronium and neutral pion. Appendix. Double-reversion as a c-number.

#1. Introduction

The aim of this Part is to give a systematic exposition of the q-number theory of space-inversion (mirage), time-reversal (reversion) and charge conjugation. As such, the present article is a review report on this field of problems which has attracted growing attention of theoreticians in recent years. However, besides being a review article, this paper is the first attempt of exposition which from the outset is based on full recognition of the fact that there exist hermitian or unitary operators which without being c-numbers can commute with all the known physical quantities. Existence of such operators was previously pointed out by the author, and the operator of double reversion was shown to be an example of such operators.[1] This fact was used by Wick, Wightman and Wigner to introduce what they called superselection rule.[1] In this paper, the superselection rule is not only given thorough investigation and natural generalization, but its basic idea is invoked at many places of the entire paper.

Although the idea of superselection is indispensable for a clear understanding of symmetry problems, and its discovery originated from a consideration of symmetry problems, the superselection itself can be proven without the help of space-time or charge symmetry. In Appendix, it will even be shown that a modified formalism of spinor field is possible, in which the operator of double reversion becomes a c-number.

In Part I, reversibility (invariance for reversion), reflectibility (invariance for mirage) and inversibility (invariance for space-and-time inversion) have been formally defined and assumed to hold whenever necessary. This Part II provides the proofs of these invariance principles, together

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[1] S. Watanabe, Phys.Rev.81, 1008(1951). This paper will hereinafter be referred to as (R). See, in particular, discussion in connection with Eqs. (4.25) and (8.15) of (R). See also, G.C.Wick,A.S.Wightman and E.P.Wigner, Phys. Rev. 88, 101(1952).

with the proof of charge-conjugation-invariance. In some particular cases, these invariance principles are conversely used to determine the correct prescription for mixing interaction types. Combinations of two kinds of symmetry, such as reversion and charge-conjugation, are also given due consideration. The so-called "charge-symmetry", which consists in interchange of neutron with proton, is not considered in this paper.

In Sec. 2 of this part, the spinor is introduced according to Cartan's method, for this method is certainly the best suited for the c-number theoretical treatment of symmetry problems. This short introduction about the spinor is intended to fill theoretical lacunae left in the current expositions on spinors. The ten different "kinds" of spinors previously introduced by the author are also systematically explained. However, it will presently become clear that this classification of spinors is no longer of great importance in the field theoretical application of spinors. This is essentially due to the fact that the physical content of the theory is left unchanged by a gauge transformation of all the spinors involved. Specification of the "kind" of a spinor has sometimes been associated with the law of conservation of heavy particles.[2] In this paper, however, we take a viewpoint that the "kind" of a spinor does not have much physical importance and that the physical laws such as conservation of heavy particles should rather be associated with the superselection rules.

As a result of our conformity with Cartan's method, our formulae will be written with the help of the E-matrices instead of the customary γ-matrices, which may be a little obnoxious to some readers. However, which is aesthetically more acceptable, to have $E_4\psi$ and $E_1 E_2 E_3 \psi$ respectively for time-

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[2] As many pertinent references as are known to the author are cited in each individual section in the following dealing with a specific problem. No bibliographical completeness, however, has been contrived.

reversal and 3-dimensional space-inversion, or to have $\gamma_1\gamma_2\gamma_3\psi$ and $\gamma_4\psi$ respectively for these transformations? In any event, a dictionary of translation from one language to another is attached in Sec. 2.

In Sec. 3, the unitary operators R, M, I and C, corresponding to reversion, mirage, space-and-time inversion and charge-conjugation, are introduced and their properties are investigated in great detail. Sec. 4 shows how the c-number field theory has the correct behavior for space-mirage but the wrong behavior for time-reversion, and how the q-number theory can correct the reversion property alone retaining the mirage property. It will also show how the q-number theory enables us to formulate charge-conjugation.

Sections 5, 6, 7 and 8 demonstrate the existence of the unitary operators for the various transformations, thus proving the invariance of the theory for these transformations. These operators will be given explicit operatorial expressions. Section 9 discusses the transformation properties of various tensorial quantities formed with spinors, thus substantiating some of the results anticipated without proof in Part I. Sections 10 and 11 deal with the various types of nucleon-pion interaction and nucleon-lepton interaction. It will become clear through these discussions that it is meaningless to speak of space-parity or of charge parity of a single elementary particle, except in the case of a photon or a neutral pion.

For the nucleon-pion interaction, mixture of scalar type and vector type and mixture of pseudo-vector type and pseudo-tensor type are forbidden both by reversibility and charge-conjugation-invariance. Mixture of regular tensorial quantities and pseudo-tensorial quantities is also refuted from a general ground. For the nucleon-lepton interaction, there are two general families of types: the first family is characterized by coupling of the same tensorial or pseudo-

tensorial quantities originating from nucleons and from leptons. The other family is characterized by coupling of tensorial quantities of nucleons with pseudo-tensorial quantities of leptons, and vice versa.[2] The both families are equally justifiable, but mixing of the two families is forbidden. Moreover, charge-conjugation and reversibility require certain phase-factor relations among the interaction constants in each of the **families**.

Sections 5 through 9 will show how the concepts of reversion, mirage and charge-conjugation are useful in determining the eigen-functions of each field and also in establishing relationship among eigenfunctions. The angular momentum representations for photons (Sec. 6))and for electrons (Sec. 8) may prove to be instrumental for other problems than the formal problems of symmetry.

As an illustration of the selection rules originating from space-symmetry and charge-conjugation, Sec. 13 will discuss the two-photon decay of a positronium and of a neutral meson.

It was pointed out in a previous paper by the author (the paper referred to as R in the footnote 1) that reversibility as such does not determine the commutation rules of charged fields, and that it is rather charge-conjugation that does this determination. It is not intended to emphasize this point any more in this paper, but Sections 4 and 5 will give a sketch of the underlying facts. If one had the impression that reversibility had the power of determining the statistics-type, it is only because the combination of what we call reversion in this paper and charge conjugation was used for time reversal. This fact should not be interpreted as meaning that one view point is "correct" and the other is "incorrect". What we should call time-reversal is more or less a matter of linguistic taste. The important fact is that as many modes

of invariance the theory allows for, so many laws of invariance we have. Of course some of the invariance laws can be derived from other invariance laws, but all of them are "correct." It is important to recognize this fact also with regard to space parity. The author showed in a separate paper that we can define many different mirage operators leading to many different parity values of a given state. There is no room for argument as regards which one of the different definitions is the "correct" one. Even after we have imposed upon the mirage operator M the conditions that $M^2 = 1$ and $M\Psi_{vac} = \Psi_{vac}$, there still remains ambiguity of parity value of a state in which the number of charged particles is odd. See (8.28). Any conclusion that can be drawn from either value of parity is "correct".

#2. Mathematical Preliminaries

A. Rotations and Reflections

The mathematical entity now called spinor was first introduced by Elie Cartan[3] in 1913, i.e., some fifteen years before the discovery of the relativistic wave-equation of the electron. Cartan's method is based on the fact that any congruent transformation can be considered as a product of simple reflections, although the way of this decomposition involves certain arbitrariness. For this reason, Cartan's method is best suited for discussions pertaining to inversions, while it brings about the same results pertaining to rotations as in the ordinary method.

By the simple reflection with respect to a plane passing through the origin and having the normal vector a^μ ($\mu = 1, 2, 3, 0$), an arbitrary, regular vector x^μ is transformed into

$$x'^\mu = x^\mu - 2a^\mu (a_\nu x^\nu)/(a_\kappa a^\kappa) . \qquad (2.1)$$

The metric tensor $g_{\mu\nu}$ of the Minkowski space is given in (I. 2.2)[4]. We assume hereafter that the normal vector a^μ is always normalized to 1 or -1, and we speak of a space-like reflection or time-like reflection according as

$$a_\mu a^\mu = +1 \qquad \text{or} \qquad a_\mu a^\mu = -1 . \qquad (2.2)$$

The sign of a^μ itself is arbitrary, but the sign of $a_\mu a^\mu$ has a vital meaning.

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[3] E. Cartan, Bull. Soc. Math. de France, 41, 53 (1913). See also E. Cartan, La théorie des spineurs (Hermann et Cie, Paris, 1938).

[4] (I.2.2) means formula (2.2) of the Part I of this paper.

Let the number of space-like reflections, the number of time-like reflections and the total number of reflections of both kinds in a decomposition of a given congruent transformation be denoted respectively by ν_s, ν_t and ν. The even-odd properties of these numbers are determined when the congruent transformation is given, and we have

$$\sigma_s = (-1)^{\nu_s}, \quad \sigma_t = (-1)^{\nu_t}, \quad \sigma = (-1)^{\nu}, \tag{2.3}$$

where σ_s, σ_t and σ are given in (I.2.8), (I.2.9) and (I.2.10). It can be shown that, by a suitable choice of simple reflections, ν can be made equal to or less than 4.

The general congruent transformations can be classified into four categories:[5] $(\mathfrak{A})$ $\sigma_s = +1$, $\sigma_t = +1$; $(\mathfrak{B})$ $\sigma_s = -1$, $\sigma_t = -1$; $(\mathcal{L})$ $\sigma_s = -1$, $\sigma_t = +1$; $(\mathcal{T})$ $\sigma_s = +1$, $\sigma_t = -1$. The invariance of the theory for $(\mathfrak{A})$ (Lorentz transformations and space-rotations) is well established. All the transformations belonging to class $(\mathcal{L})$ can be obtained by multiplying all the transformations of class $(\mathfrak{A})$ to any single transformation of class $(\mathcal{L})$, say,

$$(x, y, z, t) \rightarrow (-x, -y, -z, t) . \tag{2.4}$$

All the transformations of class $(\mathcal{T})$ can be obtained by multiplying all the transformations of class $(\mathfrak{A})$ to any single transformation of class $(\mathcal{T})$, say,

$$(x, y, z, t) \rightarrow (x, y, z, -t) . \tag{2.5}$$

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[5] See Appendix of reference (R), footnote 1 of the present paper.

And finally all the transformations of class $(\mathcal{L})$ can be obtained by multiplying all the transformations of class $(\mathfrak{a})$ to any single transformation of class $(\mathcal{L})$, say, the product of (2.4) and (2.5). Thus the study of invariance for the general congruent transformations reduces to that of the transformations given in (2.4) and (2.5).

We shall here insert illustrations to show how a spatial rotation and a Lorentz transformation can be decomposed into simple reflections. Take two purely spatial unit vectors: $\mathbf{a}$ and $\mathbf{b}$. ($a^0 = b^0 = 0$). We perform reflections first with regard to $\mathbf{a}$ and then with regard to $\mathbf{b}$:

$$\left.\begin{aligned} \mathbf{x}' &= \mathbf{x} - 2\mathbf{a}(\mathbf{a}\cdot\mathbf{x}), \\ \mathbf{x}'' &= \mathbf{x}' - 2\mathbf{b}(\mathbf{b}\cdot\mathbf{x}'). \end{aligned}\right\} \tag{2.6}$$

Putting $$[\mathbf{a}\times\mathbf{b}] = \sin\frac{\varphi}{2}\cdot\mathbf{c}\;, \quad (\mathbf{a}\cdot\mathbf{b}) = \cos\frac{\varphi}{2}\;, \tag{2.7}$$

where $\mathbf{c}$ is a unit vector, we can easily deduce from (2.6)

$$\begin{aligned} \mathbf{x}'' = \mathbf{x} &+ (1-\cos\varphi)[\mathbf{c}\times[\mathbf{c}\times\mathbf{x}]] \\ &+ \sin\varphi\,[\mathbf{c}\times\mathbf{x}]\;. \end{aligned} \tag{2.8}$$

As a simple geometrical consideration will show, this transformation (2.8) is nothing but a rotation by angle φ about the axis $\mathbf{c}$. Suppose we perform furthermore a simple reflection with regard to $\mathbf{c}$, then we obtain

$$\begin{aligned} \mathbf{x}''' &= \mathbf{x}'' - 2\mathbf{c}(\mathbf{c}\cdot\mathbf{x}) \\ &= -\mathbf{x} - (1+\cos\varphi)[\mathbf{c}\times[\mathbf{c}\times\mathbf{x}]] + \sin\varphi\,[\mathbf{c}\times\mathbf{x}]\;. \end{aligned} \tag{2.9}$$

This reduces to (2.4) for $\varphi = \pi$, agreeing with the fact that the total mirage is the product of a simple reflection with regard to any direction in space and a rotation by 180° about this direction.

In a similar way, let us take two space-like unit vectors:

$$\left.\begin{aligned} a^\mu &= (\cosh\alpha, 0, 0, \sinh\alpha), \\ b^\mu &= (\cosh\beta, 0, 0, \sinh\beta), \end{aligned}\right\} \tag{2.10}$$

and perform simple reflection first with regard to a^μ and then with regard to b^μ. Then, the product of these two simple reflections becomes

$$\left.\begin{aligned} x'^1 &= \cosh\varphi\, x^1 + \sinh\varphi\, x^0, \\ x'^0 &= \sinh\varphi\, x^1 + \cosh\varphi\, x^0, \end{aligned}\right\} \tag{2.11}$$

with

$$\varphi = 2(\beta - \alpha). \tag{2.12}$$

(2.11) is just an ordinary Lorentz transformation with a relative velocity in x^1-direction of magnitude v :

$$\sinh\varphi = v/\sqrt{1-v^2}. \tag{2.13}$$

The interchange of a^μ with b^μ changes the sign of the relative velocity.

B. Spinors

The basic matrices E_μ ($\mu = 1, 2, 3, 0$) are defined by

$$E_\mu E_\nu + E_\nu E_\mu = 2g_{\mu\nu}. \tag{2.14}$$

They must have at least four columns and four rows to satisfy (2.14). The "contravariant" E's are defined by

$$E^\mu g_{\mu\nu} = E_\nu. \tag{2.15}$$

The matrix E_4 defined by

$$E_4 = iE^0 = -iE_0 \tag{2.16}$$

will be occasionally used, but usually the index μ will be supposed to run over (1,2,3,0) instead of (1,2,3,4). It is also to be noted that E_5 defined

by

$$E_1 E_2 E_3 E_0 = iE_5 \qquad (2.17)$$

anticommute with the four E's and $E_5^2 = 1$. The change of factor i into -i in (2.17) will change what Eddington would call chirality, but it does not affect the ordinary four dimensional theory.

There exist two important matrices J and K, which transform the basic E's respectively into their hermitian conjugates and their transposes.[6] J is a necessary instrument to construct the "adjoint" spinors, and K is closely connected with the charge-conjugation. Their properties are, more precisely,

$$J^{-1} E_\mu J = -\bar{E}_\mu \;, \; J^{-1} E_5 J = -\bar{E}_5 \;, \; \bar{J} = J; \qquad (2.18)$$

$$K^{-1} E_\mu K = -E_\mu^T \;, \; K^{-1} E_5 K = +E_5^T \;, \; K^T = -K, \qquad (2.19)$$

where the bar means the hermitian conjugate, and T means the transpose. Between J and K exists a relation:

$$\bar{K} = -J^T K^{-1} J. \qquad (2.20)$$

There exist sets of E's such that E_1, E_2, E_3, E_4 ($\equiv -i E_0$) and E_5 are all hermitian (E_0 : antihermitian):

$$E_\mu = \bar{E}^\mu \;, \quad E_5 = \bar{E}_5 . \qquad \text{(hermitian system)} \; (2.21)$$

In a hermitian system, we can take

$$J = J^{-1} = \bar{J} = E_4 = -iE_0 , \qquad \text{(hermitian system)} \; (2.22)$$

and we have

$$\bar{K} = K^{-1}. \qquad \text{(hermitian system)} \quad (2.23)$$

Furthermore, among the hermitian systems, there exist such sets of E's, that E_0 is antisymmetric and that three out of E_1, E_2, E_3, E_5 are

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[6] The J used here is i-times or minus i-times the J used in (R). For the derivation of the theorems concerning K and J, see W. Pauli, Ann. Inst. Henri Poincare, 6, 137 (1936).

symmetric and the remaining one is antisymmetric. If we take E_2 for instance as this antisymmetric one, then we have the following rule:

$$\left.\begin{array}{ll} E_1, E_3, E_0, E_5 & \text{have real elements,} \\ E_2 & \text{has imaginary elements.} \end{array}\right\} \text{(special)} \qquad (2.24)$$

In such a system we can take

$$J = -iE_0 ; \qquad K = E_1 E_3 . \qquad \text{(special)} \qquad (2.25)$$

If we take E_5 as the antisymmetric one, we have the following rule:

$$\left.\begin{array}{ll} E_1, E_2, E_3, E_0 & \text{have real elements,} \\ E_5 & \text{has imaginary elements.} \end{array}\right\} \text{(special)} \qquad (2.26)$$

In such a system, we can take

$$J = K = -iE_0 . \qquad (2.27)$$

As will presently be seen, we have to adopt a hermitian system in order to make the normalization of spinors in the ordinary 3-dimensional space feasible. We shall not use particular representations of J and K, except in the last stage of calculations, in order to keep the physical implications of J and K clear. As to the special systems (2.24) (2.25), they are mentioned only to compare the E-system to the usually adopted representations of the α's and the γ's, and will not be used at all in this paper.

We now introduce a 4-4-matrix representation of the entire congruent group. Any congruent transformation can be expressed as the result of a series of simple reflections characterized by normal vectors: $a^{\mu}_{(1)}$, $a^{\mu}_{(2)}$, . . ., $a^{\mu}_{(\nu)}$, whereby the order in which the reflections appear must be respected. Corresponding to this transformation, we introduce a

matrix:

$$S = A_\nu \cdots A_2 A_1 , \tag{2.28}$$

with

$$A_\tau = a^\mu_{(\tau)} E_\mu . \tag{2.29}$$

It should be noted that

$$AA = a^\mu a^\nu \tfrac{1}{2}(E_\mu E_\nu + E_\nu E_\mu) = a_\mu a^\mu . \tag{2.30}$$

Therefore,

$$S^{-1} = \sigma_t \cdot A_1 A_2 \cdots A_\nu . \tag{2.31}$$

It is easy[7] to show that the S's defined by (2.28) are faithful representations of the congruent group, but they are by definition two-valued since the sign of each a^μ is arbitrary. From the defining properties of J and K follow

$$J^{-1} S J = \sigma_5 \bar{S}^{-1} , \tag{2.32}$$

$$K^{-1} S K = \sigma_5 S^{T-1} . \tag{2.33}$$

The spinor ξ is defined as the representation vector of the S's, i.e.,

$$\xi' = S \xi . \tag{2.34}$$

The adjoint spinor has to be defined by a transformation rule which contains S^{-1}, so that it may cancel with the transformation of ξ. This could be done, in view of (2.32) and (2.33), either by the use of J or K. However, in order to provide the possibility of positive definite normalization of spinors, we have to choose the former alternative. Thus, the adjoint spinor $\xi^\times$ is defined by

$$\xi^\times = \bar{\xi} J^{-1} . \tag{2.35}$$

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[7] See Cartan, loc. cit.

where the bar means the complex conjugate in c-number theory and hermitian conjugate in q-number theory. The star will be reserved to designate exclusively the complex conjugate. In virtue of (2.32) and (2.34), the transformation rule of $\xi^{\times}$ becomes

$$\xi'^{\times} = \sigma_3 \xi^{\times} S^{-1}. \tag{2.36}$$

Now, relation (2.33) can be exploited to derive a physically important theorem. Namely, if ξ_1 and $\xi_1^{\times}$ transform according to (2.34) and (2.36), then ξ_2 and $\xi_2^{\times}$ defined by

$$\xi_2 = \xi_1^{\times} K \quad , \qquad \xi_2^{\times} = \xi_1 K^{-1} \tag{2.37}$$

transform, on account of (2.33), again according to the same rules, (2.34) and (2.36). That ξ_2 and $\xi_2^{\times}$ defined by (2.37) are connected by (2.35) can be shown by (2.20). We shall see that transformation (2.37) provides the possibility of charge-conjugation.

The "normalization" of spinors is usually done by requiring

$$\int \xi^{\times} i E^{0} \xi \, dx\,dy\,dz = +1 , \tag{2.38}$$

which is possible in a hermitian E-system, because, on account of (2.22), (2.38) becomes

$$\int \bar{\xi} \xi \, dx\,dy\,dz = +1 .$$

We shall have to use constantly quantities of the type $\xi^{\times} O \xi$ where O is some operator with Dirac indices. It should always be kept in mind that if

$$Q = \xi^{\times} O \xi = \bar{\xi} O' \xi , \tag{2.39}$$

this means that

$$O' = -iE_0 Q \quad \text{or} \quad O = -iE_0 O'. \tag{2.40}$$

If O' is hermitian, then Q will be real in the c-number theory.

It may be well to introduce here a set of simplifying notations:

$$\left.\begin{aligned} E_{\mu\nu} &= (1/2) \sum P(\alpha\beta)\, E_\alpha E_\beta = E_\mu E_\nu - E_\nu E_\mu , \\ E_{\mu\nu\varkappa} &= (1/6) \sum P(\alpha,\beta,\gamma)\, E_\alpha E_\beta E_\gamma , \\ E_{\mu\nu\varkappa\lambda} &= (1/24) \sum P(\alpha,\beta,\gamma,\delta)\, E_\alpha E_\beta E_\gamma E_\delta , \end{aligned}\right\} \quad (2.41)$$

where the P's are sign-functions such that, for instance, $P(\alpha,\beta,\gamma)$ is + 1 when (α,β,γ) is an even permutation of $(\mu,\nu,\varkappa)$, and −1 when (α,β,γ) is an odd permutation of $(\mu,\nu,\varkappa)$.

We shall now give the transformation properties of various tensorial quantities that can be built with the two spinors ψ and φ obeying the same transformation rules (2.34) and (2.36):

$$\left.\begin{aligned} \Lambda &= i\psi^\times E_5 \varphi , & \Gamma &= \psi^\times \varphi , \\ \Lambda_\mu &= i\psi^\times E_\mu \varphi , & \Gamma_\mu &= i\psi^\times E_5 E_\mu \varphi , \\ \Lambda_{\mu\nu} &= \psi^\times E_5 E_{\mu\nu} \varphi , & \Gamma_{\mu\nu} &= i\psi^\times E_{\mu\nu} \varphi \end{aligned}\right\} \quad (2.42)$$

The factor i is inserted in four of those quantities so that the complex conjugate of each quantity becomes, in c-number theory, identical with the original quantity with ψ and φ interchanged. It is easy to show by the help of (2.34) and (2.36) that the "kinds" of tensors, defined in Part I, of these quantities are as follows:

$$\left.\begin{aligned} &\Lambda ,\ \Lambda_\mu ,\ \Lambda_{\mu\nu} && \text{second kind} \\ &\Gamma ,\ \Gamma_\mu ,\ \Gamma_{\mu\nu} && \text{third kind} \end{aligned}\right\} \quad (2.43)$$

$\Lambda_{\mu\nu}$ and $\Gamma_{\mu\nu}$ are complementary to each other on account of (2.17). See(I.2.13). The assignment of kinds given in (2.43) will be no longer valid in the q-number theory; see Table III, # 9.

To avoid repetition, let us prove only that Λ_μ is a second kind pseudo-vector. Taking an arbitrary regular vector x^μ, we build a matrix

$$X = x^\mu E_\mu .$$

If x^μ undergoes the transformation (2.1), X will undergo the transformation:

$$X' = x'^\mu E_\mu = x^\mu E_\mu - 2\sigma_t a^\mu E_\mu (a_\kappa x^\kappa)$$

$$= X - \sigma_t A(AX + XA) = -\sigma_t AXA ,$$

where A is given by (2.29). More generally, if x^μ undergoes a series of simple reflections, X will obviously transform according to

$$\begin{aligned} X' &= (-1)^\nu \sigma_t A_\nu \cdots A_1 X A_1 \cdots A_\nu \\ &= \sigma S X S^{-1} . \end{aligned} \qquad (2.44)$$

Now take the product of Λ_μ with x^μ and call it C:

$$C = \Lambda_\mu x^\mu , \qquad (2.45)$$

then this will transform, on account of (2.34), (2.35) and (2.44),

$$C' = i\,\psi'^\times E_\mu \varphi' x'^\mu = \sigma_s \sigma\, \psi^\times E_\mu \varphi\, x^\mu , \qquad (2.46)$$

showing that C is a second kind scalar. Then, in virtue of the quotient rule of tensor calculus and of the product rule of Part I, we conclude that Λ_μ is a second kind pseudo-vector.

(2.44) can also be written, with the help of $a^{\mu}_{\cdot\nu}$ (I.2.1), as

$$a^{\mu}_{\cdot\nu}\, x^\nu E_\mu = \sigma S\, x^\mu E_\mu S^{-1}. \qquad (2.47)$$

Since x^ν is arbitrary, we obtain

$$E_\mu a^{\mu}_{\cdot\nu} = \sigma S E_\nu S^{-1}. \qquad (2.48)$$

Relation (2.48) enables us to compare the E-system with the customary α-system and γ-system. More precisely, the transformation rules for class $\mathfrak{a}$ alone are not sufficient to determine the unique correspondence between the E-system and the γ-system. If we assume for the γ-system the transformation rule (including inversions):

$$\gamma_\mu a^{\mu}{}_{.\nu} = S\gamma_\nu S^{-1}, \tag{2.49}$$

then we obtain the following correspondence:

$$\left.\begin{aligned}
E_1 &= i\gamma_5\gamma_1 = -i\alpha_2\alpha_3\beta = \sigma_1\rho_3\\
E_2 &= i\gamma_5\gamma_2 = -i\alpha_3\alpha_1\beta = \sigma_2\rho_3\\
E_3 &= i\gamma_5\gamma_3 = -i\alpha_1\alpha_2\beta = \sigma_3\rho_3\\
E_0 &= iE_4 = -\gamma_5\gamma_4 = -i\alpha_1\alpha_2\alpha_3\beta = -i\rho_2\\
E_5 &= \gamma_5 = i\alpha_1\alpha_2\alpha_3 = -\rho_1 = i\rho_2\rho_3
\end{aligned}\right\} \tag{2.50}$$

$$\left.\begin{aligned}
&\gamma_1 = iE_1E_5\,,\ \gamma_2 = iE_2E_5\,,\ \gamma_3 = iE_3E_5\,,\ \gamma_4 = E_0E_5\,,\ \gamma_5 = E_5\\
&\alpha_1 = E_0E_1\,,\ \alpha_2 = E_0E_2\,,\ \alpha_3 = E_0E_3\,,\ \beta = E_0E_5
\end{aligned}\right\} \tag{2.51}$$

$$\left.\begin{aligned}
\xi^{\times} &= -i\bar{\xi}E_0 = -i\xi^{\dagger}\gamma_5\\
\xi^{\dagger} &= \bar{\xi}\gamma_4 = i\xi^{\times}E_5
\end{aligned}\right\} \tag{2.52}$$

$$\left.\begin{aligned}
\Lambda &= \psi^{\dagger}\varphi\,, & \Gamma &= -i\psi^{\dagger}\gamma_5\varphi\\
\Lambda_\mu &= i\psi^{\dagger}\gamma_\mu\varphi\,, & \Gamma_\mu &= i\psi^{\dagger}\gamma_\mu\gamma_5\varphi\\
\Lambda_{\mu\nu} &= -i\psi^{\dagger}\gamma_{\mu\nu}\varphi\,, & \Gamma_{\mu\nu} &= i\psi^{\dagger}\gamma_{\mu\nu}\gamma_5\varphi
\end{aligned}\right\}(\mu,\nu=1,2,3,4) \tag{2.53}$$

From this correspondence, it can be seen that the hermitian E's result in hermitian α's and hermitian γ's. Furthermore, the customary representation of α's in which only α_2 has imaginary elements corresponds

to the special system (2.24),(2.25). The γ-system in which only γ_4 and γ_5 have imaginary elements corresponds to the special system (2.26),(2.27). If we translate the charge-conjugation (2.37) into the language of the γ-system with the help of (2.50) and (2.52), we get

$$\xi_2 = -i\,\xi_1^{\dagger} E_5 K$$

Comparing this with Schwinger's definition[8] of charge-conjugation matrix C:

$$\xi_2 = -\,\xi_1^{\dagger} C \tag{2.54}$$

we get $C = i\,\gamma_5 K$ and, in particular, in the system (2.26),(2.27),

$$C = -\gamma_4 \quad , \qquad \xi_2 = \bar{\xi}_1 \tag{2.55}$$

We shall now briefly discuss the spinor transformations corresponding to the illustrations given in (2.8) and (2.11). The transformation matrices S corresponding to (2.6) are, according to (2.29),

$$A_a = a^{\alpha} E_{\alpha} \;, \qquad A_b = b^{\alpha} E_{\alpha} \qquad (\alpha = 1,2,3) \tag{2.56}$$

Then the S for (2.8) will be

$$\left.\begin{aligned} &S = A_b A_a = -\sum_{\gamma} \sin\frac{\varphi}{2}\; c^{\gamma} E_{\alpha\beta} + \cos\frac{\varphi}{2} \\ &(\alpha,\beta,\gamma : \text{cyclic in } 1,2,3) \end{aligned}\right\} \tag{2.57}$$

In particular, when $\varphi = \pi$, (2.57) becomes

$$S_1 = -\left(E_{12} C_3 + E_{23} C_1 + E_{31} C_2\right) \tag{2.58}$$

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[8] J. Schwinger, Phys.Rev., 74, 1439 (1948).

The simple reflection with regard to $\mathbf{C}$ is

$$S_2 = E_1 C_1 + E_2 C_2 + E_3 C_3 \tag{2.59}$$

Finally, the product of these two, which is the mirage, is given by

$$S_3 = S_2 S_1 = -E_1 E_2 E_3 \tag{2.60}$$

These S_1, S_2, S_3 will play a certain role in the angular momentum representation of a spinor field. It should be noted with regard to (2.57) that if we continuously increase φ up to $\varphi = 2\pi$, S becomes -1 instead of $+1$, which is another manifestation of the two-valuedness of S.

Next for the transformations given in (2.10), we have

$$\left.\begin{aligned} A_1 &= E_1 \cosh\alpha + E_0 \sinh\alpha , \\ A_2 &= E_1 \cosh\beta + E_0 \sinh\beta , \end{aligned}\right\} \tag{2.61}$$

and the resultant of these, which is a Lorentz transformation,

$$S = A_2 A_1 = \cosh(\beta - \alpha) - E_1 E_0 \sinh(\beta - \alpha) . \tag{2.62}$$

The transformations considered in (2.56) through (2.62) pertain to the spinors which do not depend on the position in the space-time. Usually the spinor represents a certain field, and therefore should undergo the transformation of the argument (x, y, z, t) of ξ besides the spinor transformation considered above. As is always the case in the problems of this kind, two "pictures" are possible: either the vectors are fixed while the coordinates are shifted or the vectors are shifted while the coordinates are fixed. Taking, for definiteness, the first picture, let

us denote the transformation in question by an abstract symbol $\mathcal{T}$. The same space-time point will be given coordinates x^μ and x'^μ in the old and the new frame of reference, in such a way that $x' = \mathcal{T}x$. At the same time the field quantity F will become $\mathcal{T}F = F'$. Therefore the field quantity F in the new description is

$$F'(x') = \mathcal{T}F(x) = \mathcal{T}F(\mathcal{T}^{-1}x'). \tag{2.63}$$

As an illustration, let us apply this formula to the transformation (2.8), (2.57):

$$\begin{aligned} \xi'(\mathbf{x}') = \Big(-\sum_\gamma \sin\frac{\varphi}{2}\, C^\gamma E_{\alpha\beta} + \cos\frac{\varphi}{2}\Big) \times \\ \xi\,\big(\mathbf{x}' + (1-\cos\varphi)[\mathbb{C}\times[\mathbb{C}\times\mathbf{x}']] - \sin\varphi\,[\mathbb{C}\times\mathbf{x}']\big), \end{aligned} \tag{2.64}$$

where the sign of φ is opposite to that used in (2.8) since we have to take $\mathcal{T}^{-1}$ instead of $\mathcal{T}$ in the argument of ξ. In particular, if $\mathbb{C}$ is oriented in the z-direction,

$$\xi'(x',y',z') = \Big(-\sin\frac{\varphi}{2}E_{12} + \cos\frac{\varphi}{2}\Big)\xi\,(\cos\varphi\; x' + \sin\varphi\, y', -\sin\varphi\, x' + \cos\varphi\, y', z').$$

For the infinitesimal rotation $d\varphi$, we get then

$$\begin{aligned} [d\xi'(x',y',z')/d\varphi]_{\varphi=0} &= y'(d\xi/dx') - x'(d\xi/dy') - \tfrac{1}{2}E_{12}\xi \\ &= -i\,(L_3 + \tfrac{1}{2}\sigma_3)\,\xi\;, \end{aligned} \tag{2.65}$$

where L_3 and σ_3 are the z-components of the familiar operators of orbital angular momentum and spin angular momentum:

$$L_3 = -i(x\,\partial/\partial y - y\,\partial/\partial x)\;, \quad \sigma_3 = -i\,E_1E_2\;. \tag{2.66}$$

C. Various Kinds of Spinors

Just as the four kinds of tensors (#2, Part I) are equally justified representations, the four kinds of spinors[9]

$$\left.\begin{array}{ll} \xi' = S\xi\,, & \eta' = \sigma S\eta\,, \\ \zeta' = \sigma_t S\zeta\,, & \omega' = \sigma_5 S\omega \end{array}\right\} \qquad (2.67)$$

are equally justified spinors. The tensorial quantities (2.42) made out of two spinors belonging to the same kind have the same transformation properties as have been given with regard to the ξ-spinor. If ψ and φ of (2.42) belong to different kinds, then the transformation properties of the tensorial quantities are modified by the σ's involved in ψ and φ, just like the product rule of #2, Part I.

However, (2.67) is not the only possible analogue of pseudo-tensors for spinors. Actually, the three groups, each with two elements, $(\sigma=1, \sigma=-1)$, $(\sigma_t=1, \sigma_t=-1)$ and $(\sigma_5=1, \sigma_5=-1)$, are respectively isomorphic to the three factor groups, $(\mathcal{A}+\mathcal{B},\ \mathcal{C}+\mathcal{D})$, $(\mathcal{A}+\mathcal{C},\ \mathcal{B}+\mathcal{D})$ and $(\mathcal{A}+\mathcal{D},\ \mathcal{B}+\mathcal{C})$ of the congruent group since $\sigma^2 = \sigma_t^2 = \sigma_5^2 = 1$. However, the spinor representation is basically two-valued, and therefore -1 as well as $+1$ can serve as the identity transformation. Thus, we can equally well take $(\sqrt{\sigma}=\pm 1, \sqrt{\sigma}=\pm i)$, $(\sqrt{\sigma_t}=\pm 1, \sqrt{\sigma_t}=\pm i)$ and $(\sqrt{\sigma_5}\pm 1, \sqrt{\sigma_5}\pm i)$ as the representation of the above given factor groups. This consideration leads to the following new kinds of spinors:

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9. See Appendix, reference (R).

$$\mu' = \sqrt{\sigma}\, S \mu\,, \quad \nu' = \sqrt{\sigma_t}\, S \nu \quad, \varkappa' = \sqrt{\sigma_s}\, S \varkappa \tag{2.68}$$

where the square root of -1 can be either +i or -i, thus (2.68) actually contains six different kinds of spinors. The possibilities (2.68) were first indicated by the author[10] in connection with a 5-dimensional theory, but it applies obviously to the 4-dimensional theory, too. It can easily be seen that Yang-Tiomno's A- and B-types[11] belong to our (2.68) and C- and D-types belong to our (2.67). Our classification is more detailed than Yan-Tiomno's because classes $\mathcal{S}$ and $\mathcal{T}$ are taken into consideration.

We should not spend more space here for these different kinds of spinors, because, in q-number theory, we shall be able to deal with the possible phase changes in reversion and mirage without specifying in advance the kinds of the spinors involved.

D. Transpose Operators

In the main body of this paper, we shall have to deal constantly with transpose operators. The transpose O^T of an operator O can of course be defined in the matrix form:

$$O^T_{ij} = O_{ji} = (\bar{O}_{ji})^* . \tag{2.69}$$

The operation of transposition itself is not an invariant operation for an arbitrary unitary transformation. However, we shall see in the next section that, in spite of this, a certain operation involving transposition acquires an invariant meaning.

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[10] S. Watanabe, Sci.Pap. Inst.Phys.Chem.Research(Tokyo)39,157(1941)

[11] C.N.Yang and J.Tiomno, Phys.Rev. 79, 495(1950)

For instance, the operator $O=\partial/\partial x$ has the matrix elements:

$$(x'|O|x'') = \delta'(x'-x''). \tag{2.70}$$

Thence, its transpose is given by

$$(x'|O^T|x'') = \delta'(x''-x') = -\delta'(x'-x''). \tag{2.71}$$

Therefore, $O=\partial/\partial x$ is an antisymmetric operator in the x-representation. (2.71) shows that O^T here means the differentiation of a function standing to the left of this operator. This agrees with the more elementary definition of the transpose operator:

$$Of = fO^T. \tag{2.72}$$

In the same way, $p_x = -i\partial/\partial x$ is also an antisymmetric operator in the x-representation, because it is hermitian and has imaginary matrix elements. We shall often use an operator Ω_x which is defined by

$$\Omega_x f(x) = f(-x). \tag{2.73}$$

For this operator, we have

$$(x'|\Omega_x|x'') = (x'|\Omega_x^T|x'') = \delta(x'+x''), \tag{2.74}$$

i.e., in the x-representation,

$$\Omega_x = \Omega_x^T \ (= \bar{\Omega}_x = \Omega_x^{-1}). \tag{2.75}$$

For an operator which involves both space-time coordinates and the Dirac indices, we have to take the transpose with respect to both of these variables. For instance, if O is given by[12]

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[12] P.A.M.Dirac, The Principles of Quantum Mechanics(Oxford University Press, London, 1947) third edition, p.267, Eq.(40).

$$I = L_\alpha E_\alpha + E_0 E_5 \quad (\alpha = 1,2,3) , \tag{2.76}$$

with $$L_\alpha = -L_\alpha^T = -i\left[x_\beta(\partial/\partial x_\gamma) - x_\gamma(\partial/\partial x_\beta)\right] \quad (\alpha,\beta,\gamma\text{: cyclic in } 1,2,3) , \tag{2.77}$$

then we can write with the help of (2.19)

$$I^T = K^{-1} I K . \tag{2.80}$$

E. Redundancy of State-Functions

It is well-known that the state function Ψ has an inherent arbitrariness of phase-factor, i.e., Ψ and

$$e^{i\alpha}\Psi \tag{2.81}$$

represent physically the same state, where α is an arbitrary real constant. However, it seems to be generally overlooked[13] that α can be a particular function of the occupation number of spinor particles involved in Ψ. For instance, we can take as $\exp(i\alpha)$ the following function:

$$\left.\begin{aligned} & e^{i\alpha} = \Delta^{-1} = \bar{\Delta} = \Delta^T = \Delta \equiv \prod_i (-1)^{N_i} = \prod_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \text{with} \quad & \Delta^2 = 1 \end{aligned}\right\} \tag{2.82}$$

where the index i should run all over the spinor eigen-states. If Ψ is known to represent a state with an even (odd) number of spinor particles, then $\Delta\Psi$ will be $+\Psi$ ($-\Psi$). But in general, we have to leave Δ as an operator. Ψ and $\Delta\Psi$ will represent the same state, if Δ commutes with any arbitrary physical quantity, because, if so, these

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[13] This fact was noticed by the author in relation to Eqs. (4.25)(8.15), reference (R).

two state-functions will have the same expectation values for all the physical quantities.

The commutability of Δ with any physical quantity is a consequence of the fact that any physical quantity must involve an even number of spinors. It has usually the form $\psi^{\times} O \varphi$, but in some cases (e.g., pair creation interaction energy) $\psi^{\times} O \varphi^{\times}$ may also happen, where $\psi^{\times}$ is linear in creation operators $\bar{g}$ and φ is linear in annihilation operators g. g and $\bar{g}$ have the forms:

$$\left.\begin{aligned} g &= \left[\Pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \bar{g} &= \left[\Pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}\right\} \qquad (2.83)$$

or their transforms by an unitary transformation. The factor represented by Π is the well-known Wigner-Jordan factor. Then, we have obviously

$$\Delta g \Delta = -g \,, \qquad \Delta \bar{g} \Delta = -\bar{g} \qquad (2.84)$$

As a result any expression which contains an even number of spinors commute with Δ.

We shall later see that Δ is not the only unitary operator that commutes with all the known physical quantities. We shall in general write W for any operator that has this property. A more general consideration on this line will be given in our section on the superselection rule.

#3. Operators for Reversion, Mirage, Inversion and Charge-Conjugation.

We use the interaction "picture" throughout this paper, less for the purpose of actual calculations of specific problems in this picture, but rather in order to have at hand formulae applicable for both the Heisenberg and Schrodinger pictures. We can indeed obtain the formulae for the latter two pictures from those of the interaction picture only by including the entire Hamiltonian either in the "free" Hamiltonian or the interaction Hamiltonian of the interaction picture.

The time development of the state function $\Psi(t)$ is governed by the interaction Hamiltonian $\mathbb{H}(t)$ and expressed by

$$\Psi(t_2) = U(t_2,t_1)\Psi(t_1)\,, \tag{3.1}$$

with

$$\left.\begin{aligned} dU(t_2,t_1)/dt_2 &= -i\,\mathbb{H}(t_2)U(t_2,t_1)\,, \\ dU(t_2,t_1)/dt_1 &= +i\,U(t_2,t_1)\,\mathbb{H}(t_1)\,, \end{aligned}\right\} \tag{3.2}$$

and

$$U(t_1,t_1) = 1\,, \quad U^{-1}(t_2,t_1) = \overline{U}(t_2,t_1) = U(t_1,t_2). \tag{3.3}$$

The time-development of the physical quantity $Q(t)$ is governed by the free Hamiltonian $\mathbb{H}_0$ and expressed by

$$Q(t_2) = U_0^{-1}(t_2,t_1)\,Q(t_1)\,U_0(t_2,t_1)\,, \tag{3.4}$$

with

$$dU_0(t_2,t_1)/dt = -i\,\mathbb{H}_0 U(t_2,t_1)\,, \quad U(t_1,t_1) = 1\,. \tag{3.5}$$

According to the definitions of reversed phenomenon, miraged phenomenon and inverted phynomenon given in #3, Part I, they are characterized by

$$Q'(-t) = \rho_R\,Q(t)\,, \quad Q'(-x) = \rho_M\,Q(x)\,, \quad Q'(-x,-t) = \rho_I\,Q(x,t). \tag{3.6}$$

See (I.2.18), (I.2.19) and (I.2.21). In quantum physics, we have to reinterpret (3.6) in terms of expectation values. Thus, the reversed phenomenon $\Psi_R(t)$, miraged phenomenon $\Psi_M(t)$ and inverted phenomenon $\Psi_I(t)$ of an original phenomenon $\Psi(t)$ should be defined by

$$(\Psi_R(-t), Q(x,-t)\Psi_R(-t)) = \rho_R\,(\Psi(t), Q(x,t)\Psi(t)), \qquad (3.7)$$

$$(\Psi_M(t), Q(-x,t)\Psi_M(t)) = \rho_M\,(\Psi(t), Q(x,t)\Psi(t)), \qquad (3.8)$$

$$(\Psi_I(-t), Q(-x,-t)\Psi_I(-t)) = \rho_I\,(\Psi(t), Q(x,t)\Psi(t)), \qquad (3.9)$$

where ρ_R, ρ_M and ρ_I are given in Tables II, III, IV and V, Part I.

We add to these sign-functions another sign function ρ_C for the charge-conjugation. ρ_C is $+1$ for all the "mechanical" quantities and -1 for all the "electromagnetic" quantities. Among the various quantities mentioned in Table V, Part I, the following are "electromagnetic": electric charge, magnetic charge, current, electromagnetic potentials, electromagnetic field strengths, electromagnetic moment. All the rest are "mechanical." Using ρ_C thus defined, the charge-conjugate phenomenon $\Psi_C(t)$ should be defined by

$$(\Psi_C(t), Q(x,t)\Psi_C(t)) = \rho_C\,(\Psi(t), Q(x,t)\Psi(t)). \qquad (3.10)$$

We can now define reversibility, reflectibility, etc. as follows: If $\Psi_R(t)$, $\Psi_M(t)$, $\Psi_I(t)$ or $\Psi_C(t)$, defined in (3.7) through (4.10), is a solution of (3.1)(3.2) on condition that $\Psi(t)$ is a solution, then we speak of reversibility, reflectibility, inversibility or charge-invariance. Actually, we gave in Part I a slightly different definition of reversibility, etc. According to this definition, reversibility holds if the transition probability from an arbitrary Ⓗ to an

arbitrary Ω, say, is equal to the transition probability from the reversed state of Ω to the reversed state of Θ. We shall presently see that the definition given above entails automatically this last definition.

We now introduce four time-independent, unitary operators which are useful tools to prove the four invariances in question. Reversion operator R, mirage operator M, inversion operator I and charge-conjugation operator C are defined by

$$Q(x,-t) = \rho_R \, (R^{-1} Q(x,t) R)^T, \tag{3.11}$$

$$Q(-x,t) = \rho_M \, (M \; Q(x,t) M^{-1}), \tag{3.12}$$

$$Q(-x,-t) = \rho_I \, (I^{-1} Q(x,t) I)^T, \tag{3.13}$$

$$Q(x,t) = \rho_C \, (C \; Q(x,t) C^{-1}). \tag{3.14}$$

In the quantized field theory, the field-strengths appearing in Q are q-numbers, and the transposition T refers to the Q as a q-number. For instance,

$$(R^{-1} \psi^{\times} O \psi R)^T = R^T \psi^T O^T \psi^{\times T} R^{-1T}, \tag{3.15}$$

where T on O has the meaning of transposition discussed in #2.D. The left-hand side of (3.15) should not be equated to the negative of the right-hand side on the ground that ψ and $\psi^{\times}$ anticommute. The anticommutability of ψ and $\psi^{\times}$ is implicitely taken care of by the matrix representations of ψ and $\psi^{\times}$.

It is true that the operation of transposition is not invariant for a unitary transformation. For this reason, one may think that definition (3.11) has no physical meaning. However, this is not the case, for we

have not yet determined the transformation rule of R for a unitary transformation. Take a time-independent[14] unitary transformation V, by which Q and Ψ are transformed into

$$Q' = V^{-1} Q V \quad , \quad \Psi' = V^{-1} \Psi . \tag{3.16}$$

Then we shall have again

$$Q'(x,-t) = \rho_R \, (R'^{-1} Q'(x,t) R')^T , \tag{3.17}$$

in the primed system if we take

$$R' = V^{-1} R V^* , \tag{3.18}$$

where $V^* = \bar{V}^T$. Thus, we assume (3.18) to be the transformation rule of R for the time-independent unitary transformation V. We have the same rule for I.

We shall now show that R, M, I and C, if they exist, enable us to build Ψ_R, Ψ_M, Ψ_I and Ψ_C, defined in (3.11) through (3.15), from a given Ψ:

$$\left.\begin{aligned} &\Psi_R(-t) = \Psi^*(t) R \quad , && \Psi_M(t) = M \Psi(t) , \\ &\Psi_I(-t) = \Psi^*(t) I \quad , && \Psi_C(t) = C \Psi(t) . \end{aligned}\right\} \tag{3.19}$$

We can of course insert arbitrary phase-factors in these equations, but for the moment we assume them to be included in R, etc. To avoid repetitions, let us prove only that Ψ_R defined by (3.19) in fact satisfies (3.7). By the use of (3.19), the left-hand side of (3.7) becomes

$$(R^T \Psi^*(t), Q(-t) R^T \Psi^*) = (\Psi^*(t) , \bar{R}^T Q(-t) R^T \Psi^*(t)) .$$

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[14] For a time-dependent unitary transformation, see towards the end of this section.

On account of (3.11), this is equal to

$$= \rho_R (\Psi^*(t),\ \bar{Q}^*(t)\Psi^*(t)) = \rho_R (\Psi(t),\ \bar{Q}(t)\Psi(t))^*.$$

This is equal to the right-hand side of (3.7). The proof runs in a similar way for M, I and C.

It should be noted that if Ψ and Ψ_R undergo transformation (3.16) and R undergoes transformation (3.18), then (3.19) holds again in the primed system.

Now, it is clear that reversibility, reflectibility, inversibility and charge invariance will be established if we can prove that Ψ_R, Ψ_M, Ψ_I and Ψ_C defined in (3.19) are solutions of (3.1)(3.2) on condition that Ψ is a solution. We shall now show that this is in fact the case. Take two transformation functions $U(t,0)$ and $U(0,-t)$ which are, according to (3.2), given by

$$\left.\begin{aligned} dU(t,0)/dt &= -i\,\mathbb{H}(t)\,U(t,0), \\ dU(0,-t)/d(-t) &= +i\,U(0,-t)\,\mathbb{H}(-t), \end{aligned}\right\} \qquad (3.20)$$

with $U(0,0) = 1$

Their transposes are then determined by

$$\left.\begin{aligned} dU^T(t,0)/dt &= -i\,U^T(t,0)\,\mathbb{H}^T(t), \\ dU^T(0,-t)/d(-t) &= +i\,\mathbb{H}^T(-t)\,U^T(0,-t), \end{aligned}\right\} \qquad (3.21)$$

with $U^T(0,0) = 1$.

Applying R on (3.20), we get (on accout of time-independence of R)

$$\left.\begin{aligned} d\,R^{-1}U(t,0)R/dt &= -i\,\mathbb{H}^T(-t)\,R^{-1}U(t,0)R \\ d\,R^{-1}U(0,-t)R/d(-t) &= +i\,R^{-1}U(0,-t)R\,\mathbb{H}^T(t) \end{aligned}\right\} \qquad (3.22)$$

with $$R^{-1}U(0,0)R = 1 \,,$$

because $(\rho_R = +1 \text{ for energy})$

$$\mathbb{H}(-t) = (R^{-1}\mathbb{H}(t)R)^T. \qquad (3.23)$$

Comparing (3.22) with (3.21), we obtain

$$R^{-1}U(t,0)R = U^T(0,-t) \,, \quad R^{-1}U(0,-t)R = U^T(t,0) \,. \qquad (3.24)$$

Combining the two equations of (3.24), we get[15] for $U(t,-t) = U(t,0)\,U(0,-t)$

$$U^T(t,-t) = R^{-1}U(t,-t)R \,. \qquad (3.25)$$

In a similar way

$$\left.\begin{aligned} U(t,-t) &= M \;\; U(t,-t)M^{-1}, \\ U^T(t,-t) &= I^{-1}U(t,-t)I \,, \\ U(t,-t) &= C \;\; U(t,-t)\,C^{-1}. \end{aligned}\right\} \qquad (3.26)$$

It should be noted that the two equations (3.24) are equivalent to each other if $R^T R^{-1}$ commute with U. We shall discuss this condition more in detail later.

Once (3.25) is established, it is an easy matter to show that if

$$\Psi(t) = U(t,-t)\,\Psi(-t) \,, \qquad (3.27)$$

then $$\Psi_R(t) = U(t,-t)\,\Psi_R(-t) \,. \qquad (3.28)$$

In fact, on account of (3.19) and (3.27), we have

$$\Psi_R(t) = \Psi^*(-t)R = \Psi^*(t)\,U(t,-t)R \,, \qquad (3.29)$$

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[15] There is a misprint in the corresponding equation (4.27) in (R).

which, in virtue of (3.25), equals

$$= \Psi^*(t) R\, U^T(t,-t) = U(t,-t)\Psi_R(-t). \tag{3.30}$$

Q.E.D. The proof runs the same way for the other operators.

What has been shown is, in a word, that if R exists to satisfy (3.11), then reversibility is guaranteed. We shall show in the following sections that such R in fact exists.

Incidentally, by the use of

$$R^{-1} U_0(t,0) R = U_0^T(0,-t) \quad \text{etc.}, \tag{3.31}$$

which is a consequence of

$$H_0 = (R^{-1} H_0 R)^T, \tag{3.32}$$

we can rewrite (3.11) in a simpler way:

$$Q(x,0) = \rho\, (R^{-1} Q(x,0) R)^T. \tag{3.33}$$

The derivation is similar to our argument regarding (3.20) through (3.24).

Another important consequence of (3.25) is the alternative definition of reversibility used in Part I. Suppose two arbitrary states Θ and Ω are given. In one process, we consider the transition probability from Θ at $t = -t_1$, to Ω at $t = +t_1$. In the other process we consider the transition probability from the reversed state Ω_R of Ω at $t = -t_1$ to the reversed state Θ_R of Θ at $t = +t_1$:

$$\Theta_R = \Theta^* R, \quad \Omega_R = \Omega^* R. \tag{3.34}$$

The first transition probability is determined by the transition matrix

$$(\Omega, U(t_1,-t)\Theta)$$

and the second by

$$(\Theta_R, U(t_1,-t_1)\Omega_R) = (\Theta^*, R^{T-1}\, U(t_1,-t_1)\, R^T\Omega^*).$$

On account of (3.25), we can rewrite this last expression as

$$= (\Theta, \vec{U}(t_1,-t_1)\Omega)^* = (\Omega, U(t_1,-t_1)\Theta).$$

Thus,
$$(\Omega, U(t,-t)\Theta) = (\Theta_R, U(t,-t)\Omega_R). \qquad (3.35)$$

From this follows, to use the notation of Part I, that

$$P(\Theta \to \Omega, 2t) = P(\Omega_R \to \Theta_R, 2t). \qquad (3.36)$$

In a similar way, we obtain

$$\left.\begin{aligned}(\Omega, U(t,-t)\Theta) &= (\Omega_M, U(t,-t)\Theta_M)\\ &= (\Theta_I, U(t,-t)\Omega_I)\\ &= (\Omega_C, U(t,-t)\Theta_C).\end{aligned}\right\} \qquad (3.37)$$

It should now be noted that the four operators defined by (3.11) through (3.14) have two types of basic arbitrariness:

$$S \to e^{i\alpha} S, \qquad (3.38)$$

and
$$S \to W S. \qquad (3.39)$$

where S stands for any one of the four operators, and W is supposed to satisfy

$$W Q W^{-1} = Q \qquad (3.40)$$

for any known physical quantity Q, as has been explained towards the end of the last section. For our study of symmetry properties of physical laws, however, we need to pick any one S that satisfies the defining equation; the arbitrariness expressed by (3.39) can be relegated to a separate study

of operators satifying (3.40).

Suppose then that one such R satisfying (3.11) is obtained and that its phase-factor in the sense of (3.38) is arbitrarily fixed. Then (3.19) has to be written more generally

$$\Psi_R(-t) = e^{i\alpha}\Psi^*(t)R, \tag{3.41}$$

with an arbitrary real α for the reversed state, satisfying (3.7) and (3.28). If we take two different values of α in (3.34), we get of course an arbitrary phase-factor appearing in (3.35), but (3.36) is still true. For M, I, and C in (3.37), the situation is the same.

Now, if $\Psi_R(-t)$ is the reversed state of $\Psi(t)$, then $\Psi(-t)$ must be the reversed state of $\Psi_R(t)$. Therefore, we must have

$$\Psi(-t) = e^{i\beta}\Psi_R^*(t)R. \tag{3.42}$$

Combining (3.41) and (3.42), we obtain

$$\Psi(t) = e^{i(\beta-\alpha)}R^TR^{-1}\Psi(t) \tag{3.43}$$

for any Ψ. (3.43) represents the effect of a "double reversion."

(3.43) shows that R^TR^{-1} is either a c-number $e^{i(\alpha-\beta)}$, or a q-number for which Ψ is an eigenfunction corresponding to eigen-value $e^{i(\alpha-\beta)}$. Such a q-number must obviously be of the type (3.40), exemplified by Δ (2.82), since otherwise Ψ and $R^TR^{-1}\Psi$ would be physically distinguishable from each other.

Whatever R^TR^{-1} may be, we cannot exclude the possible existence of self-reversed processes in Nature. The simplest example is the vacuum-state. For such a state α and β must be the same, and we obtain

$$\Psi_{self\,R}(t) = R^T R^{-1} \Psi_{self\,R}(t) . \tag{3.44}$$

If $R^T R^{-1}$ is a c-number, then (3.44) gives

$$R^T R^{-1} = 1 \qquad \text{or} \qquad R^T = R . \tag{3.45}$$

If $R^T R^{-1}$ is, for instance, a multiple of Δ, i.e., if $R^T R^{-1} = a\Delta$, then (3.44) shows the existence of an eigenstate of $R^T R^{-1}$ for which the eigenvalue is + 1. Since Δ has eigenvalues ± 1, a must be ± 1. In particular, since Δ operated on the vacuum-state Ψ_{vac} must give +1, due to the definition (2.82):

$$\Delta \Psi_{vac} = \Psi_{vac} \tag{3.46}$$

and since Ψ_{vac} is a self-reversed state, we conclude that $a = 1$, or

$$R^T R^{-1} = \Delta . \tag{3.47}$$

It should be noted that the transformation (3.18) leaves the condition (3.45) unchanged. It also leaves the condition (3.47) unchanged provided V commutes with Δ. This commutability may be assumed as quite a general rule since Δ must be unobservable before as well as after the trasformation. The transformation (3.38) can be considered as an especially simple case of (3.18). The transformation (3.39) transforms $R^T R^{-1}$ as follows:

$$R^T R^{-1} \rightarrow R^T R^{-1} (R W^T R^{-1} W^{-1}) \tag{3.48}$$

i.e., if $R = W R W^*$, then $R^T R^{-1}$ remains unchanged.

We shall later see that for the spinor fields we get actually (3.47) and not (3.45). No transformation W is discovered that makes $R W^T R^{-1} W^{-1} = \Delta$,

so the the transformation (3.48) would result in (3.45). We shall instead show later the if we use 8-component spinors we can realize (3.45). For the boson fields, we obtain (3.45) in a quite natural fashion.

A by-product of this consideration is that (3.44) is satisfied only if $\Delta = +1$, i.e., the self-reversed states must involve even numbers of spinor particles. It should however be kept in mind that this conclusion is derived in a purely field theoretical point of view, i.e., the entire physical system is inclusively represented by Ψ.

As for the operators M and C, if we perform a double-mirage and a double-charge-conjugation, in a similar way to (3.41) and (3.42), we obtain

$$\Psi(t) = e^{i(\alpha+\beta)} M^2 \Psi(t) \quad , \quad \Psi(t) = e^{i(\alpha+\beta)} C^2 \Psi . \tag{3.49}$$

This shows that M^2 and C^2 must be a c-number or a W, in the sense of (3.40). If the former is the case, we have to put $M^2 = a$, and $C^2 = b$, where a and b are some constants of absolute value unity. But this a and b can be transformed easily into unity by the transformation (3.38), which does not change anything essential except that the eigenvalues are renamed. Therefore we can in this case put in general,

$$M^2 = 1 \quad , \quad C^2 = 1 \, , \tag{3.50}$$

i.e.,

$$M = M^{-1} = \bar{M} \quad , \quad C = C^{-1} = \bar{C} \, . \tag{3.51}$$

In contrast to the case of $R^T R^{-1}$, the conditions (3.50) on M^2 and C^2 determine the numerical factors of M and C. We shall see that we can find in a quite natural way M and C which satify (3.50) in both spinor and boson cases. However, it should be noted that the transform-

ation (3.39) leads to

$$M^2 \rightarrow (WMWM^{-1})\ M^2 \qquad (3.52)$$

$$C^2 \rightarrow (WCWC^{-1})\ C^2 \qquad (3.53)$$

the right sides of which are not necessarily c-numbers even if the original M and C obey (3.50). As a matter of fact, we can easily find a W which commutes with M and C and whose square is not a c-number. Under these conditions, the right-hand sides of (3.52) and (3.53) become W^2 which is again an operator of the type (3.40). Actually, if we use the general expression of M satisfying (3.12), we are led to the expression of M^2 which is not a c-number, and this fact was used to discover a new W.[16] However such M can be brought back to an operator satisfying (3.50) by a transformation of the type (3.39). Therefore, we can always assume (3.50) and study the W's (3.40) as a separate problem.

Coming back to (3.49), if we take as Ψ a self-miraged state or a self-charge-conjugate state, we obtain, under the assumption (3.50),

$$e^{2i\alpha} = 1 \quad , \quad e^{i\alpha} = \pm 1 . \qquad (3.54)$$

In other words, an operator satisfying (3.50) will have eigenvalues +1 and −1, corresponding to which there will be two groups of eigenfunctions, dividing self-miraged or self-charge-conjugate states into two groups. This is the origin of the even and odd parity with regard to space-symmetry and charge-interchange.

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[16] S. Watanabe, Phys. Rev. in press.

We should now spend a few words for combinations of two operators introduced in (3.11) through (3.14). As a representative of the various combinations, let us first consider the time-reversal considered by some authors,[17] which is actually the combination of the reversion in our sense and the charge-conjugation. According to this point of view, the reversed state Ψ'_R is defined by

$$(\Psi'_R(-t), Q(x,-t)\Psi'_R(-t)) = \rho_R \rho_C (\Psi(t), Q(x,t)\Psi(t)), \qquad (3.55)$$

instead of (3.7). Such a state can be considered either as the reversed state in our sense of the charge-conjugate state or the charge-conjugate state of the reversed state in our sense. Accordingly, the operator R' playing the role of R in this point of view, i.e., the operator such that $\Psi'_R(-t) = \overset{*}{\Psi}(t) R$, will be given by either

$$R' = C^{-1} R \qquad \text{or} \qquad R' = R C^{T}. \qquad (3.56)$$

Since these two operators must have physically the same effect, we have to put in general

$$C^{-1} R = W R C^{T} \qquad (3.57)$$

where W may be a c-number or an operator of the type (3.40). If we adopt $R' = C^{-1} R$ and $C^2 = 1$, the double time-reversal in this modified point of view will be represented by

$$R'^{T} R'^{-1} = R^{T} R^{-1} W^{-1} \qquad (3.58)$$

where W is the one used in (3.57).

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[17] Called standpoint (I) in paper (R). See, for instance, J. Schwinger, Phys. Rev. 82, 914 (1951).

As a matter of fact, exploiting the arbitrariness still disposable within the conditions (3.47) and (3.50), we can adjust C and R in such a way that W in (3.57) becomes an operator with a wide range of arbitrariness, including $W = $ c-number as well as $W = \Delta$. As a result, R', in contrast to R, can satisfy $R'^T R'^{-1} = \Delta$ as well as $R'^T R'^{-1} = 1$, among other possibilities.[18]

Another important combination is one of reversion operator and mirage operator, which should result in the total inversion operator I :

$$I = M^{-1} R \quad \text{or} \quad I = R M^T. \tag{3.59}$$

Since R' and I can thus be obtained from the basic three operators: R, C, and M, we shall not discuss R' and I any further in this paper. If reversibility and charge-invariance hold, then the invariance for the modified time-reversal will hold. If reversibility and space-invariance hold, then inversibility will hold.

Before closing this section, let us briefly consider the effect on R of a time-dependent unitary transformation:

$$Q'(t) = V^{-1}(t) Q(t) V(t) \quad , \quad \Psi'(t) = V^{-1}(t) \Psi(t). \tag{3.60}$$

We can re-establish (3.11) and (3.19) in the primed system by taking

$$R' = V^{-1}(t) R V^{T-1}(-t) . \tag{3.61}$$

We can pass from the interaction picture to the Schrödinger picture by putting

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[18] See discussions connected with Eqs. (6.5) and (6.19) in paper (R).

$$V^{-1}(t) = U_0(t,0) . \tag{3.62}$$

The reversion operator in the Schrödinger picture then becomes

$$R' = U_0(t,0)\, R\, U_0^{T-1}(0,-t) = R , \tag{3.63}$$

on account of (3.31). This shows that we can use the same R in the Schrödinger pisture.

In a similar way, we can show by the use of (3.24) that the reversion operator in the Heisenberg picture is the same as in the interaction picture. By the same token, M, C, and I remain unchanged in all the three pictures.

#4. Passage from C-Number Theory to Q-Number Theory

Before discussing the symmetry properties of physical quantities in the q-number theory, it may be well to survey the situation in the c-number theory and to see in what respects the c-number theory encounters difficulties. As is well-known, the c-number theory is analogous to the Heisenberg picture in the sense that the time-development is attributed solely to the time-dependence of the physical quantities.

The problem in the c-number theory lies in the following situation. The field variables u (which may be tensorial or spinorial) are supposed to have certain transformation properties. The physical quantities Q, such as spin-density, electric current, etc., are expressed in terms of field variables u. By the transformation properties attributed to the u's, we can therefore determine the sign function ρ' (for reversion, mirage, etc.) of each Q. On the other hand, due to the physical consideration given in Part I, each Q has a definite ρ. The problem is then to see whether the ρ' determined by the transformation properties of the field variables u actually coincide with the ρ determined by physical requirements. We shall presently see that this is not always the case.

In reference to Eq. (2.63) of #2.B, let us consider $\mathfrak{T}$ as representing reversion $\mathfrak{R}$, mirage $\mathfrak{M}$ or inversion $\mathfrak{I}$. In these cases, we have: $\mathfrak{T}^{-1} = \mathfrak{T}$, or $\mathfrak{T}^2 = 1$. Of course, for spinors the effect of $\mathfrak{T}$ has an ambiguity of sign; as a result, $\mathfrak{T}^2 = \pm 1$. But, the physical quantities being of even order in spinors, this ambiguity does not affect our argument and we can assume $\mathfrak{T}^2 = 1$ without loss of generality.

For the field variable u , (2.63) becomes

$$u'(x') = \sigma u(\sigma^{-1} x') . \qquad (4.1)$$

This can be interpreted as follows: Let u(x) and u'(x) represent the original process and its transformed (i.e., reversed, etc.) process. Then u(x) and u'(x) are connected by (4.1).

Now each physical quantities Q(x) has an expression in terms of the u(x) and some operator which may also depend on x. Therefore we can write

$$Q(x) = Q[u(x) ; x] . \qquad (4.2)$$

For the transformed process, we have to take u' at σx and to transform the explicit x in (4.2) into σx . Thus we have to compare with (4.2) the following quantity:

$$Q[u'(\sigma x) ; \sigma x] ,$$

which, in virtue of (4.1), becomes

$$Q[\sigma u(x) ; \sigma x] .$$

The operation σ has been shifted from the argument x to the tensorial or spinorial function u. The ϱ' which was in question in the foregoing must then be given by

$$\left.\begin{aligned} Q[\sigma u(x) ; \sigma x] &= \varrho'_{\sigma} Q[u(x) ; x] , \\ \text{or} \qquad Q[\sigma u(\sigma^{-1} x) , x] &= \varrho'_{\sigma} Q[u(\sigma^{-1} x) ; \sigma^{-1} x] . \end{aligned}\right\} \qquad (4.3)$$

The second equation is obtained from the first just renaming σx as x. The first one shows that ϱ'_{σ} can be determined simply considering the

tensorial or spinorial transformation of u and the transformation of the coordinates explicitly involved in Q.

We shall soon see that the ρ' thus determined is not necessarily the same as the ρ given in tables of Part I. The physical quantities can be classified into "mechanical" and "electromagnetic". In the field theory, mechanical quantities are primarily derived from the "free" Lagrangean density, by the well-known procedure. For instance, if the Lagrangean density is a regular scalar, the energy-momentum tensor $T^{\mu\nu}$ and the angular momentum tensor $M^{\mu\nu\kappa}$ are regular tensors according to this procedure. We have shown in Part I that the energy momentum tensor should belong to the regular kind. As to the angular momentum tensor, we notice that the complementary vector, in the sense of (I.2.12): $M_\mu = \frac{1}{6}\epsilon_{\mu\nu\kappa\lambda} M^{\nu\kappa\lambda}$ becomes a first kind pseudo-vector. We have explained in #3, Part I, that the angular momentum (including spin), if represented as a vector, should behave like a first kind pseudo-vector. This shows that if the Lagrangean density is a regular scalar, all the rest of the mechanical quantities will become what they should be. In a similar way, if the current-density appearing in the interaction Lagrangean density is a second kind pseudo-vector, all the rest of electromagnetic quantities will behave as they should. For this reason, we shall limit our discussion to the free Lagrangean and the current-density. This will be the case also in our q-number discussion. If it is desired, one can check the results for individual physical quantities, but we shall skip this discussion in this paper.[19]

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[19] There are discussions of this kind in (R). See also #9 of this paper.

Let us first consider the free Lagrangean of tensorial fields v, i.e., pion-field and electromagnetic field. This term depends on v through a combination of the type $\bar{v}v$; or $\bar{v}_\mu v^\mu$, and it contains an operator which is a regular scalar, such as $(\partial/\partial x_\nu)(\partial/\partial x^\nu)$. Now, whatever the kind of the field v may be, the combination of the type $\bar{v}v$ or $\bar{v}_\mu v^\mu$ is a regular scalar on account of the product rule, #2, Part I. Therefore, the free Lagrangean of tensorial fields is a regular scalar. Next the current-density of a tensorial field is typified by

$$s^\mu(x) = ie\left[(\partial\bar{v}(x)/\partial x_\mu)\,v(x) - \bar{v}(x)(\partial v(x)/\partial x_\mu)\right], \qquad (4.4)$$

which refers to a scalar or pseudo-scalar pion field. $\bar{v}v$ being a regular scalar, (4.4) is a regular vector. This is, however, not the desired transformation of the current-density, which should be a second kind pseudo-vector. In other words ρ'_M is all right, but ρ'_R is wrong. This is also true for the current due to a vector or pseudo-vector pion-field.

Next, for a spinor field, the free Lagrangean has the form:

$$\mathcal{L}(x) = -\frac{1}{2}\psi^\times(x)\left[E_\mu(\partial/\partial x_\mu) - E_\mu(\overleftarrow{\partial/\partial x_\mu}) + 2miE_5\right]\psi(x). \qquad (4.5)$$

Now, since $\psi^\times E_\mu\psi$ and $\psi^\times E_5\psi$ belong to the second kind, (2.43), and $\partial/\partial x_\mu$ to the regular kind, $\mathcal{L}$(x)(4.5) is a second kind scalar. In other words, $\rho'_M = \rho_M$ but $\rho'_R = -\rho_R$, in the c-number theory. On the other hand, the current density due to a charged spinor field:

$$s^\mu(x) = ei\,\psi^\times(x)E^\mu\psi(x) \qquad (4.6)$$

is already a second kind pseudo-vector, (2.43), as it should be.

As a conclusion, we can say the following in the c-number theory. As far as mirage is concerned, ρ'_M is exactly what it should be, $\rho'_M = \rho_M$, for all the quantities. For reversion, the mechanical quantities of tensorial fields and the electromagnetic quantities of spinorial fields have the right signs, $\rho'_R = \rho_R$, but the electrical quantities of tensorial fields and the mechanical quantities of spinorial fields have the wrong signs, $\rho'_R = -\rho_R$.

Regarding the charge-conjugation, we can see from (4.4) that the interchange of υ and $\bar{\upsilon}$ will result in a reversal of sign of the current. Thus, we can define the c-number transformation for charge conjugation by

$$\upsilon \to e^{i\alpha} \bar{\upsilon} , \qquad \bar{\upsilon} \to e^{-i\alpha} \upsilon . \tag{4.7}$$

It should be noted already at this stage that by (4.7) a representative term in the free Lagrangean will be transformed like

$$(\partial \bar{\upsilon}/\partial x_\mu)(\partial \upsilon/\partial x^\mu) \longrightarrow (\partial \upsilon/\partial x_\mu)(\partial \bar{\upsilon}/\partial x^\mu),$$

showing that, if we adopt the Fermi-statistics in q-number theory, the transformation (4.7) will result in an undesired sign-change of energy.

For the charged spinor field it is suggested by (4.7) that ψ may probably be transformed to a multiple of ψ^x. Since we have seen that ψ and $\psi^x K$ have the same transformation rule for the congruent group, it would be a natural choice to take

$$\psi \to e^{i\alpha} \psi^x K , \qquad \psi^x \to e^{-i\alpha} \psi K^{-1}, \tag{4.8}$$

as the c-number transformation rule for charge conjugation. However (4.8) would transform (4.6) into

$$-ei\psi K^{-1}E^{\mu}K\psi^{x} = ei\psi E^{\mu T}\psi^{x} \tag{4.9}$$

on account of (2.19). (4.9) is, in the c-number theory, equal to (4.6). Thus (4.9) does not change the sign of the current-vector. However, if we adopt the Fermi-statistics in q-number theory, (4.9) will become the negative of (4.6), "except for a c-number additional term". We shall later see that we can drop this last restrictive clause by adopting the so-called Heisenberg prescription.

Now, we shall sketch, in anticipation, what will be done in the q-number theory to remedy what was wrong in the c-number theory and to retain what was right in the c-number theory. Take, for instance, Eq. (3.12), where the matrix M is, of course, supposed to operate on the field variables. The left-hand side of (3.12) means, to use the notation of this section (4.2),

$$Q(\mathfrak{M}x) = Q[u(\mathfrak{M}x); \mathfrak{M}x],$$

and the right-hand side means

$$Q[M\,u(x)M^{-1}; x],$$

i.e., (3.12) is equivalent to

$$Q[M\,u(x)M^{-1}; x] = \rho_M Q[u(\mathfrak{M}x); \mathfrak{M}x]. \tag{4.10}$$

Since $\rho_M' = \rho_M$, the second equation of (4.3) becomes

$$Q[\mathfrak{M}u(\mathfrak{M}x); x] = \rho_M Q[u(\mathfrak{M}x), \mathfrak{M}x], \tag{4.11}$$

having the same right side as (4.10). Therefore all we need do in the q-number theory is to put

$$M\,u(x)M^{1} = \mathfrak{M}u(\mathfrak{M}x). \tag{4.12}$$

Thus, for instance, since we know that the effect of $\mathcal{M}$ on a spinor is essentially multiplication by E_{123}, we get from (4.12)

$$M\psi(x,t)M^{-1} = e^{i\alpha}E_{123}\psi(-x,t). \tag{4.13}$$

If we want to be more general, we could also put

$$M\psi(x,t)M^{-1} = e^{i\alpha}E_{123}W\psi(-x,t).$$

However, this is nothing but the combination of (4.13) with a transformation of the type

$$W'\psi W'^{-1} = W\psi \quad , \quad W'QW'^{-1} = Q, \tag{4.14}$$

where W' is again an operator of the type (3.40), which can be studied separately from space-symmetry.

As for reversion, we have to note that the right side of (3.11) involves the transposition. As a result, the effect of R on u must contain the transposition ($u \rightarrow u^T$) besides the transformation that could be inferred from a comparison with the c-number theory. First, regarding the tensorial field, ρ'_R is wrong only for the electromagnetic quantities. We can remedy this by assuming that the effect of R is the combination of the c-number transformation for $\mathcal{R}$ and the c-number transformation for $\mathcal{L}$ (4.7):

$$R^{-1}u(x)R = \mathcal{R}\mathcal{L}\,u^T(\mathcal{R}x). \tag{4.15}$$

As $\mathcal{L}$ does not change the sign of mechanical quantities, the correct sign ρ'_R of the free Lagrangean will be retained by (4.15). Moreover, since $\mathcal{L}$ (4.7) interchanges once $\mathcal{V}$ and $\overline{\mathcal{V}}$ and since (3.11) also interchanges $\mathcal{V}$ and $\overline{\mathcal{V}}$, as can be seen from (3.15), the tranformation (4.15) will serve the purpose without any specific assumption as to the statistics.

Suppose we apply the same procedure (4.15) to the spinor field, namely let us assume the effect of R is essentially the combination of the

c-number transformation of ψ for reversion and the charge-conjugation (4.8). Then, on account of the c-number reversion $\mathcal{L}(x)$ changes the sign ρ', but the effect of (4.8) will be such that $(\partial/\partial x_\mu)$ is interchanged to $(\overleftarrow{\partial/\partial x_\mu})$. Thus, it will change the sign of the first two terms of (4.5). Besides, since $K^{-1}E_5K = E_5^T$, while $K^{-1}E_\mu K = -E_\mu^T$, the effect of (4.8) will be such that $\psi^\times E_5\psi$ changes the sign, in contrast to (4.9). Thus, such a combination will correct the sign ρ_R' of (4.5). On the other hand, (4.8) does not change the sign of (4.6). Therefore the right sign of the current will be retained. In this argument, it is understood that, since (3.11) involves the interchange of the factors ψ and $\psi^\times$ and (4.8) also interchanges these factors, we need not invoke a specific assumption regarding the statistics. This shows that the desired effect of R would be

$$R^{-1}\psi(t)R = e^{i\alpha}E_0 K\psi^{\times T}(-t), \tag{4.16}$$

where E_0 stands for the c-number transformation for $\mathcal{R}$.

As regards charge-conjugation, we have already seen that (4.7) and (4.8) gives the correct signs provided that we adopt Bose-statistics for tensorial fields and Fermi-statistics for spinorial fields. Thus, we should take

$$C u C^{-1} = \mathcal{C} u, \tag{4.17}$$

where $\mathcal{C}$ is (4.7) or (4.8)

The pion-nucleon interaction plays a unique role in determining the kind of pion-field. Tensorial fields in general appear in a quadratic or belinear form in the free Lagrangean. Therefore, their sign does not affect at all the mechanical behavior of their fields. However, as has

been seen in Part I, there seems to be a sound physical reason to believe that the electromagnetic field is "observable", in the sense that its sign-change actually changes the physical situation. Thus we could meaningfully assign the second kind to the electromagnetic quantities, This determination, in essence, has been done through their interaction with the charged field, by assuming that, in a reversed state, a particle of the same charge is performing a kinematically reversed motion.

In contrast to this, there is room for doubt as to whether there is any physical meaning in assigning a kind to the pion-field. If this assignment is to be done, it must be possible only through the pion-nucleon interaction, which is the only term where the mesic field strength is standing in the first power. We could indeed determine the interaction type by experiments, and require the regular invariance of the interaction Lagrangean. However, the sign-change of the source involved in this interaction is not necessarily uniquely dtermined. For the neutral pion, the source is represented by a quantum jump from a proton state to another proton state, or from a neutron state to another neutron state. Therefore, the source has a definite sign-change for $\mathfrak{M}$ and $\mathfrak{R}$ for the neutral meson. However, for a charged pion field, the sourve represents a transition from a proton state to a neutron state, or vice versa. Therefore the sign-change of the source depends on the *relative* phase-change of proton spinor and neutron spinor for $\mathfrak{M}$ and $\mathfrak{R}$. There is so far no reason to believe that there should be any physical meaning in such a relative phase-change of proton and neutron. Therefore, in conclusion, we could say something regarding the neutral pion field, but we

cannot say anything definite regarding the changed pion field. See #10, for more details.

In passing, it should be noted that we have to include in (4.15) and (4.17) the arbitrary phase-factor exp ($i\alpha$) of (4.7) and (4.8). The reason is that, even if we obtain (4.15) and (4.17) with a fixed phase-factor in a certain gauge-reference, an arbitrary phase-factor exp($i\alpha$) will re-appear by an arbitrary gauge-transformation, because these transformations connect a field variable to its hermitain conjugate. The situation is different for M , since a gauge-transformation will leave (4.12) or (4.13) invariant. This situation is connected with the fact that the condition: M^2 =c-number actually determines the phase-factor involved in (4.12), e.g., exp($i\alpha$) in (4.13), while the conditions on C^2 and $R^T R^{-1}$ have no bearings on the phase-factors in (4.7) and (4.8). This situation will become more clear when we shall discuss the concrete cases.

#5. Pion Fields

We can describe the pion field alternatively by three hermitian field components $u^{(\rho)}$ (ρ = 1,2,3) or by one complex component v and a hermitian component $u^{(3)}$, with

$$u^{(1)} = (v + \bar{v})/\sqrt{2} \quad , \quad u^{(2)} = (v - \bar{v})/\sqrt{2}\,i \; . \tag{5.1}$$

However, the gauge-transformation (with constant phase)

$$v \rightarrow e^{i\varphi}\, v \quad , \quad \bar{v} \rightarrow e^{-i\varphi}\, \bar{v} \; , \tag{5.2}$$

is equivalent to:

$$\left.\begin{aligned} u^{(1)} &\rightarrow \cos\varphi\, u^{(1)} - \sin\varphi\, u^{(2)} \\ u^{(2)} &\rightarrow \sin\varphi\, u^{(1)} + \cos\varphi\, u^{(2)} \end{aligned}\right\} \tag{5.3}$$

resulting in mixing of $u^{(1)}$ and $u^{(2)}$. Therefore, there is no observational ground for separating the charged pion-fileds into $u^{(1)}$- field and $u^{(2)}$- field. The effect of a transformation $\mathcal{T}$, expressed in terms of independent $u^{(1)}$ and $u^{(2)}$ will then involve a danger of losing from sight the physically pertinent general features.

We shall first discuss a scalar u (of unspecified kind) and later briefly survey the case of a vector u (of unspecified kind). The free Lagrangean density and the Lagrangean density of electromagnetic interaction are, in the case of a scalar u ,

$$\begin{aligned} \mathcal{L}_u = &- (\partial\bar{v}/\partial x_\mu)(\partial v/\partial x^\mu) - \varkappa^2\, \bar{v}\, v \\ &- (1/2)(\partial u^{(3)}/\partial x_\mu)(\partial u^{(3)}/\partial x^\mu) - (1/2)\,\varkappa^2\, u^{(3)} u^{(3)} , \end{aligned} \tag{5.4}$$

$$\mathcal{L}_{Au} = A_\mu s^\mu - A_\mu A^\mu s \; , \tag{5.5}$$

where s^μ is given by (4.4) and s by

$$s = e^2\, \bar{v}\, v \, . \tag{5.6}$$

The pion-nucleon interaction will be discussed separately in #10.

The defining equations of R, M and C are (3.11), (3.12) and (3.14) applied to $\mathcal{L}_u$, s^μ and s . Since $\mathcal{L}_u$ and s must belong to the regular kind and s^μ to the second kind, we should have

$$\left.\begin{array}{llll} \rho_R = \rho_M = \rho_c = 1 & \text{for} & \mathcal{L}_u & \\ -\rho_R = -\rho_M = -\rho_c = 1 & \text{for} & s^\alpha & (\alpha = 1, 2, 3) \\ \rho_R = \rho_M = -\rho_c = 1 & \text{for} & s^0 & \\ \rho_R = \rho_M = \rho_c = 1 & \text{for} & s & \end{array}\right\} \qquad (5.7)$$

From (4.7) and (4.17), we get

$$C v C^{-1} = e^{i\alpha} \bar{v} , \quad C \bar{v} C^{-1} = e^{-i\alpha} v , \quad C u^{(3)} C^{-1} = \pm u^{(3)} . \qquad (5.8)$$

It is obvious that (5.8) satisfies the requirements (5.7) for $\mathcal{L}_u$ only if v obeys the Bose-statistics. As for s^μ , if it is written in the form given in (4.4), the assumption regarding statistics is necessary in order to satisfy (5.7). However, the c-number theory can just as well lead to the expression of s^μ :

$$s^\mu = ie[(\partial \bar{v}/\partial x_\mu) v - (\partial v/\partial x_\mu) \bar{v}] . \qquad (5.9)$$

If we use this expression for s^μ , the transformation (5.8) satisfies the requirement (5.7) for s^μ without an assumption regarding statistics. It is to be noted that (5.8) is compatible with the supposed properties of C: $C^2 = 1$, $\bar{C}C = 1$. (3.51). The transformation (5.8) also leaves the usual commutation rules unchanged.

Since the interchange of a factor in v and a factor in $\bar{v}$ results, on account of the D-function in the commutation relation, in an additional c-number term, it is desirable to replace an expression of Q by

$$Q \rightarrow (1/2) [Q + \rho_c C Q C^{-1}] , \qquad (5.10)$$

or at least to require that the expression of Q must be such that it remains

invariant by the transformation (5.10). Then we need no longer be concerned about the additional c-number term. If the quantities are written according to (5.10), charge-invariance is automatically satisfied. If we apply (5.10), for instance, to $\mathcal{L}_u$, then the expression will become such that it will vanish for the Fermi-statistics.

A similar prescription to (5.10) was first introduced by Heisenberg in relation to positron theory to avoid the zero-point charge, and (5.10) has the same effect for the υ-field.[20] For instance, the expression (4.4) of the current does not obey the Heisenberg prescription, but (5.9) does. It is well known that (5.9) leads to the vanishing zero-point charge. It is easily understandable that if the theory is exactly charge-invariant, then there should not appear any zero-point charge. For, if there would be any zero-point charge, it should be positive as well as negative.

Next, following (4.15), we can write down the relations that R should satisfy:

$$\left.\begin{aligned} (R^{-1}\upsilon(t)R)^T &= e^{i\beta}\,\bar{\upsilon}(-t) \\ (R^{-1}\bar{\upsilon}(t)R)^T &= e^{-i\beta}\,\upsilon(-t) \\ (R^{-1}u^{(3)}(t)R)^T &= \pm\, u^{(3)}(-t) \end{aligned}\right\} \qquad (5.11)$$

This satisfies (5.7) for $\mathcal{L}_u$ automatically. As for s^μ , if it is expressed as (4.4), then (5.11) automatically satisfies (5.7) for s^μ . If it is written as (5.9), we need an assumption regarding statistics to satisfy (5.7)

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[20] W. Heisenberg, Zeits. f. Physik 90, 209 and 92, 692 (1934). See also W. Pauli, Rev. Mod. Phys. 13, 203 (1941), in particular, p. 208 & p. 224.

for s^μ . However, since the c-number theory provides no reason for preference between (4.4) and (5.9), we cannot claim that reversibility has the power to determine the statistics.

It should be noted that (5.11) is compatible with the anticipated properties of R: $R^T = R$ (3.45), $\bar{R} = R^{-1}$. The transformation (5.11) also leaves the commutation rule of $\mathcal{V}$ and $\mathcal{U}^{(3)}$ unchanged.

The combination of C and R, considered in connection with (3.56), has two expressions:

$$\left.\begin{aligned} C\,(R^{-1} v(t) R)^T C^{-1} &= e^{-i(\alpha-\beta)}\, \mathcal{V}(-t) \\ (R^{-1} C\, v(t)\, C^{-1} R)^T &= e^{i(\alpha-\beta)}\, \mathcal{V}(-t) \end{aligned}\right\} \qquad (5.12)$$

The difference between these two, expressed by the unitary transformation W in (3.57), is essentially a gauge-transformation of $\mathcal{V}$, which will be discussed separately.

Analoguously to (5.10), in order to have the "exact" reversibility it is desirable to require:

$$Q(t) \to (1/2)\left[Q(t) + \rho_R (R^{-1} Q(-t) R)^T\right] . \qquad (5.13)$$

For instance, the expression formally desirable for the current would then be

$$\begin{aligned} s^\mu = (1/2)\, ie\, [\, (\partial\bar{v}/\partial x_\mu)\, v - \bar{v}\,(\partial v/\partial x_\mu) \\ + v\,(\partial\bar{v}/\partial x_\mu) - (\partial v/\partial x_\mu)\,\bar{v}\,] \end{aligned} . \qquad (5.14)$$

But, this obviously does not bring about anything new.

The above argument shows that the requirement of charge invariance determines the statistics-type of the field[21] but the requirement of reversibility does not. It is of some interest to note that we can re-state this situation saying that the compatibility of (5.10)(charge-invariance) and (5.13) (reversibility) determines the statistics. For instance, the compatibility of

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[21] W. Pauli and F. J. Belinfante, Physica 7, 177 (1940).

(4.4) and (5.9) requires the Bose-statistics. We can generalize this statement to the other quantities and also to the other fields. However, the underlying fact is that if we suitably express the physical quantities in the forms allowed by the c-number theory, reversibility alone can always be satisfied without a specific assumption regarding statistics.

If we translate the transformations (5.8) and (5.11) into the language of the hermitian $u^{(\alpha)}$, we obtain

$$\left.\begin{aligned} C u^{(1)} C^{-1} &= \cos\alpha\, u^{(1)} + \sin\alpha\, u^{(2)} \\ C u^{(2)} C^{-1} &= \sin\alpha\, u^{(1)} - \cos\alpha\, u^{(2)} \\ (R^{-1} u^{(1)}(t) R)^T &= \cos\beta\, u^{(1)}(-t) + \sin\beta\, u^{(2)}(-t) \\ (R^{-1} u^{(2)}(t) R)^T &= \sin\beta\, u^{(1)}(-t) - \cos\beta\, u^{(2)}(-t) \end{aligned}\right\} \qquad (5.15)$$

showing that it is meaningless to speak of a transformation rule for $u^{(1)}$ alone or $u^{(2)}$ alone. We can write $\mathcal{L}_u$ in two terms referring separately to $u^{(1)}$ and $u^{(2)}$, but equivalence of such an expression to (5.4) requires the Bose-assumption.

Finally the mirage operator is, in accordance with (4.12), given by

$$\left.\begin{aligned} M v(x) M^{-1} &= e^{i\gamma} v(-x) \\ M \bar{v}(x) M^{-1} &= e^{-i\gamma} \bar{v}(-x) \\ M u^{(3)}(x) M^{-1} &= \pm u^{(3)}(-x) \end{aligned}\right\} \qquad (5.16)$$

These relations are compatible with the relation $M^{-1} = \bar{M}$ and the commutation rules. However, in order to satisfy $M^2 =$ c-number, we have to adopt γ such that

$$e^{i\gamma} = \pm 1 = e^{-i\gamma} \qquad (5.17)$$

This situation makes a contrast to the case of charge-conjugation in which

$C^2 =$ c-number does not determine the phase-factor in (5.8). The double-sign in (5.17) and the double-sign in the third relation of (5.16) are of course independent of each other.

Expressing the $\mathcal{U}$-field by its Fourier expression:

$$\mathcal{V}(\mathbf{x},t)=\sum_{\mathbf{k}}(2V\omega)^{-\frac{1}{2}}\left[g_+(\mathbf{k})\exp(+-)+\bar{g}_-(\mathbf{k})\exp(-+)\right], \tag{5.18}$$

$$\mathcal{U}^{(3)}(\mathbf{x},t)=\sum_{\mathbf{k}}(2V\omega)^{-\frac{1}{2}}\left[g_3(\mathbf{k})\exp(+-)+\bar{g}_3(\mathbf{k})\exp(-+)\right], \tag{5.19}$$

with $\exp(+-)=\exp(+i\mathbf{k}\mathbf{x}-i\omega t)$, $\exp(-+)=\exp(-i\mathbf{k}\mathbf{x}+i\omega t)$ (5.20)

$$\omega=|(\mathbf{k}^2+\varkappa^2)^{\frac{1}{2}}|$$

we get

$$\left.\begin{aligned} C\,g_+(\mathbf{k})\,C^{-1} &= e^{i\alpha}g_-(\mathbf{k})\\ C\,g_-(\mathbf{k})\,C^{-1} &= e^{-i\alpha}g_+(\mathbf{k})\\ C\,g_3(\mathbf{k})\,C^{-1} &= \pm\, g_3(\mathbf{k}) \end{aligned}\right\} \tag{5.21}$$

$$\left.\begin{aligned} (R^{-1}g_+(\mathbf{k})R)^T &= e^{i\beta}\,\bar{g}_+(-\mathbf{k})\\ (R^{-1}g_-(\mathbf{k})R)^T &= e^{-i\beta}\bar{g}_-(-\mathbf{k})\\ (R^{-1}g_3(\mathbf{k})R)^T &= \pm\,\bar{g}_3(-\mathbf{k}) \end{aligned}\right\} \tag{5.22}$$

$$\left.\begin{aligned} M\,g_+(\mathbf{k})\,M^{-1} &= (\pm)\;g_+(-\mathbf{k})\\ M\,g_-(\mathbf{k})\,M^{-1} &= (\pm)\;g_-(-\mathbf{k})\\ M\,g_3(\mathbf{k})\,M^{-1} &= (\pm)'\;g_3(-\mathbf{k}) \end{aligned}\right\} \tag{5.23}$$

It is easy to show that C, R and M can be expressed in the operator forms as follows:[22]

$$C=(-1)^{\Phi}(\pm 1)^{\Theta}$$

$$\left.\begin{aligned} \Phi &= (1/2)\sum_{\mathbf{k}}\left[N_+(\mathbf{k})+N_-(\mathbf{k})-e^{+i\alpha}\,\bar{g}_+(\mathbf{k})g_-(\mathbf{k})-e^{-i\alpha}\,\bar{g}_-(\mathbf{k})g_+(\mathbf{k})\right)\\ \Theta &= \sum_{\mathbf{k}}N_3(\mathbf{k}) \end{aligned}\right\} \tag{5.24}$$

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[22] The method here used is taken from L. Wolfenstein and D.G.Ravenhall. Phys. Rev. 88, 279 (1952); see, in particular, footnote 7, p. 280.

$$R = e^{\Phi}(-1)^{\Psi}(-1)^{\Theta}$$

with
$$\left.\begin{aligned}\Phi &= \sum_{\pm}\sum_{k}(\pm i\beta N_{\pm}(k))\\ \Psi &= (1/2)\sum_{\pm}{\sum_{k}}'[N_{\pm}(k)+N_{\pm}(-k)-\bar{g}_{\pm}(k)g_{\pm}(-k)-\bar{g}_{\pm}(-k)g_{\pm}(k)]\\ \Theta &= (1/2){\sum_{k}}'[N_3(k)+N_3(-k)-(\pm)\bar{g}_3(k)g_3(-k)-(\pm)\bar{g}_3(-k)g_3(k)]\end{aligned}\right\} \quad (5.25)$$

$$M = (-1)^{\Phi}(-1)^{\Theta}$$

with
$$\left.\begin{aligned}\Phi =& (1/2){\sum_{k}}'[N_+(k)+N_+(-k)-(\pm)\bar{g}_+(k)g_+(-k)-(\pm)\bar{g}_+(-k)g_+(k)\\ &+N_-(k)+N_-(-k)-(\pm)\bar{g}_-(k)g_-(-k)-(\pm)\bar{g}_-(-k)g_-(k)]\\ \Theta =& (1/2){\sum_{k}}'[N_3(k)+N_3(-k)-(\pm)'\bar{g}_3(k)g_3(-k)-(\pm)'g_3(-k)g_3(k)]\end{aligned}\right\} \quad (5.26)$$

The arbitrary signs of the entire C and M are so chosen that $C\Psi_{vac} = M\Psi_{vac} = \Psi_{vac}$.

The summation symbol with a prime: Σ' over k means a summation such that k and $-k$ would cover the entire momentum space. For (5.25), it is assumed that g and $\bar{g}$ are expressed in the usual representation in which they have real matrix elements. $(\pm)$ and $(\pm)'$ in (5.26) refer to the same double-signs in (5.

The vector pion field (of unspecified kind) can be described by a complex vector $v^{\mu}(x,t)$ and a hermitian vector $u^{(3)\mu}(x,t)$ with $\mu = 1,2,3,0$. The transformation rules corresponding to (5.8), (5.11) and (5.16) will become

$$C v_{\mu} C^{-1} = e^{i\alpha}\bar{v}_{\mu}, \quad C u_{\mu}^{(3)} C^{-1} = \pm u_{\mu}^{(3)}, \tag{5.27}$$

$$(R^{-1} v_{\mu}(t) R)^{T} = e^{i\beta}\bar{v}^{\mu}(-t), \quad (R^{-1} u_{\mu}^{(3)}(t) R)^{T} = \pm u^{(3)\mu}(-t), \tag{5.28}$$

and
$$M v_{\mu}(x) M^{-1} = (\pm) v^{\mu}(-x), \quad M u_{\mu}^{(3)}(x) M^{-1} = (\pm)' u^{(3)\mu}(-x). \tag{5.29}$$

It should be noticed that we have in (5.28) and (5.29) covariant components on the left sides and contravariant components on the right sides. This will take care of the vectorial transformations of the field variables for $\mathcal{R}$ and $\mathcal{M}$.

The plane wave expansion of the vectorial field needs a little caution. The waves will be decomposed into transversal waves (r = 1, 2) and longitudinal waves (r = 3). The transformation rules for $\mathcal{R}$ and $\mathcal{M}$ of the annihilation operators are different for the two groups of waves. This will take care, on one hand, of the fact that the longitudinal polarization is defined by the propagation vector $\mathbb{k}$ which changes its sign for $\mathcal{R}$ and $\mathcal{M}$, and, on the other hand, of the fact that the component $\mu=0$ has the different transformation rules in (5.28) and (5.29). The relations corresponding to (5.21) (5.22) and (5.23) will be as follows:

$$\left.\begin{array}{ll} C g_{\pm r}(\mathbb{k}) C^{-1} = e^{\pm i\alpha} g_{\mp r}(\mathbb{k}), & r=1,2,3, \\ C g_{3r}(k) C^{-1} = \pm g_{3r}(k), & r=1,2,3, \end{array}\right\} \qquad (5.30)$$

$$\left.\begin{array}{lll} (R^{-1} g_{\pm r}(\mathbb{k}) R)^T &= e^{\pm i\beta} \bar{g}_{\pm r}(-\mathbb{k}), & r=1,2, \\ &= -e^{\pm i\beta} \bar{g}_{\pm r}(-\mathbb{k}), & r=3, \\ (R^{-1} g_{3r}(\mathbb{k}) R)^T &= \pm g_{3r}(-k), & r=1,2, \\ &= \mp g_{3r}(-\mathbb{k}), & r=3 \end{array}\right\} \qquad (5.31)$$

$$\left.\begin{array}{lll} M g_{\pm r}(\mathbb{k}) M^{-1} &= e^{\pm i\gamma} g_{\pm r}(-\mathbb{k}), & r=1,2;\ \gamma=0 \text{ or } \pi, \\ &= -e^{\pm i\gamma} g_{\pm r}(-k), & r=3 \\ M g_{3r}(\mathbb{k}) M^{-1} &= (\pm)' g_{3r}(-\mathbb{k}), & r=1,2, \\ &= -(\pm)' g_{3r}(-\mathbb{k}), & r=3. \end{array}\right\} \qquad (5.32)$$

It is assumed in these expressions that the positive directions of the transversal polarization, r = 1,2, are defined to be the same for both $\mathbb{k}$ and $-\mathbb{k}$.

#6. Electromagnetic Field

A. Linear Momentum Representation

In the case of electromagnetic field, the sign-functions ρ_C , ρ_R and ρ_M of the field variables are well-defined. Therefore, all we need is to show that the C, R and M defined by these sign-functions do not change the sign of the free Lagrangean and that they leave the Lorentz condition unchanged. The invariance of the interaction Lagrangean will be guaranteed, if the other fields are so adjusted that the electric current generated by these fields becomes a second kind pseudo-vector.

By the direct application of (3.11), (3.12) and (3.14) to the electromagnetic potential, we obtain

$$\left.\begin{aligned} C A_\mu C^{-1} &= -A_\mu \\ (R^{-1} A_\mu(t) R)^T &= -A^\mu(-t) \\ M A_\mu(x) M^{-1} &= -A^\mu(-x) \end{aligned}\right\} \qquad (6.1)$$

The transformation of a covariant component into the contravariant component takes care of the required transformation of a second kind pseudo-vector. We see that (6.1) is compatible with (3.45), (3.50) and (3.57) with W = 1. The transformations (6.1) leave the commutation relations of A unchanged.

The transformations (6.1) certainly make the free Lagrangean a regular scalar, and transform the Lorentz condition

$$(\partial A^\mu / \partial x^\mu)\, \Psi(t) = 0 \qquad (6.2)$$

into the corresponding equation for the charge-conjugate, reversed and miraged states.

In this subsection, we shall use the linear momentum representation for

simplicity's sake. In the next subsection, we shall use the angular momentum representation, because this representation is "compatible" with the parity operator. In the last subsection, we shall use a mixed representation mainly for the purpose of applications.

If we expand $A^{\mu}(x)$ by the plane-waves:

$$A^{\mu}(\mathbf{x},t) = \sum (2V\omega)^{-\frac{1}{2}} \left[g^{\mu}(\mathbf{k}) \exp(+-) + \bar{g}^{\mu}(\mathbf{k}) \exp(-+) \right], \qquad (6.3)$$
$$|\mathbf{k}| = \omega \quad , \quad \exp(+-) = \exp(+i\mathbf{k}\mathbf{x} - i\omega t),$$

the defining equations (6.1) become

$$\left.\begin{aligned} C\, g^{\mu}(\mathbf{k})\, C^{-1} &= -g^{\mu}(\mathbf{k}), \\ (R^{-1} g^{\mu}(\mathbf{k}) R)^{T} &= -\bar{g}_{\mu}(-\mathbf{k}), \\ M\, g^{\mu}(\mathbf{k}) M^{-1} &= -g_{\mu}(-\mathbf{k}). \end{aligned}\right\} \qquad (6.4)$$

(6.4) shows that in a representation in which g and $\bar{g}$ have real matrix elements, R and M become identical. The explicit expressions of C, R and M are then (the sign being so determined that $C\, \Psi_{vac} = M\, \Psi_{vac} = \Psi_{vac}$)

$$C = \prod_{\mu} \prod_{\mathbf{k}} (-1)^{N^{\mu}(\mathbf{k})} ; \qquad (6.5)$$

$$\left.\begin{aligned} R = M &= (-1)^{\Phi}, \\ \Phi &= (1/2) \sum_{\mathbf{k}}{}' \sum_{\alpha=1}^{3} \left[N^{\alpha}(\mathbf{k}) + N^{\alpha}(-\mathbf{k}) + \bar{g}^{\alpha}(\mathbf{k}) g^{\alpha}(-\mathbf{k}) + \bar{g}^{\alpha}(-\mathbf{k}) g^{\alpha}(\mathbf{k}) \right] \\ &\quad + (1/2) \sum_{\mathbf{k}}{}' \left[N^{0}(\mathbf{k}) + N^{0}(-\mathbf{k}) - \bar{g}^{0}(\mathbf{k}) g^{0}(-\mathbf{k}) - \bar{g}^{0}(-\mathbf{k}) g^{0}(\mathbf{k}) \right]. \end{aligned}\right\} \qquad (6.6)$$

B. Angular Momentum Representation

We consider only a pure radiation field

$$A^{0} = 0 , \qquad \operatorname{div} \mathbf{A} = 0 . \qquad (6.7)$$

A physical quantity Q derivable from $\mathbf{A}$ can be written in terms of a hermitian operator O operating on $\mathbf{A}$:

$$Q = -i\, (\dot{\mathbf{A}} \cdot O\, \mathbf{A}) , \qquad (6.8)$$

where the parentheses mean the scalar product of two spatial vectors $\dot{\mathbb{A}}$ and $\mathcal{O}\,\mathbb{A}$, and $\mathcal{O}$ is in general a tensor in the 3-dimensional space. We have to consider energy W, orbital angular momentum $\mathbb{L}$, spin angular momentum $\mathbb{S}$ and total angular momentum $\mathbb{J}$, which are respectively given by

$$\left.\begin{array}{l} W = i\frac{\partial}{\partial t}\mathbb{1} \quad ; \quad L_x = -i[y(\partial/\partial z) - z(\partial/\partial y)]\,\mathbb{1}\ , \quad \text{etc.;} \\ S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \text{ etc.;} \quad \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad \mathbb{J} = \mathbb{L} + \mathbb{S}\,. \end{array}\right\} \tag{6.9}$$

Except for $[S_x, S_y]_- = iS_z$, $[L_x, L_y]_- = iL_z$, $[J_x, J_y]_- = iJ_z$ etc., these quantities commute among themselves and with (6.7) and also with

$$\left(\Delta - \frac{\partial^2}{\partial t^2}\right)\mathbb{A} = 0\,. \tag{6.10}$$

Calling the eigen-values of W, $\mathbb{J}^2$, $\mathbb{L}^2$, J_z, respectively ω, $j(j+1)$ $\ell(\ell+1)$, and M, we denote the corresponding eigen-functions by

$$\mathfrak{u}[\omega, j, \ell, M]\,(x, y, z, t)\,, \tag{6.11}$$

where the brackets indicate their eigen-values and the parentheses their dependence on space-time. To give an explicit form to $\mathfrak{u}$, we introduce the three eigen-vectors $\mathbb{e}[\mu]$, $(\mu = -1, 0, +1)$ of S_z:

$$\mathbb{e}[1] = -(1/\sqrt{2})(\mathbb{e}_x + i\mathbb{e}_y)\ , \quad \mathbb{e}[0] = \mathbb{e}_z\ , \quad \mathbb{e}[-1] = (1/\sqrt{2})(\mathbb{e}_x - i\mathbb{e}_y)\,, \tag{6.12}$$

which satisfy

$$(\mathbb{e}^*[\mu]\cdot\mathbb{e}[\mu']) = \delta(\mu, \mu')\,. \tag{6.13}$$

Next, we introduce the normalized eigen-functions $Y_\ell^m(\theta, \varphi)$ of L_z and $\mathbb{L}^2$ which satisfy

$$\int Y_\ell^{m*}\, Y_{\ell'}^{m'} \sin\theta\, d\theta\, d\varphi = \delta(m, m')\,\delta(\ell, \ell')\,. \tag{6.14}$$

Now we can write $\mathfrak{u}$ with the help of the Clebsch-Gordon coefficients:[23]

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[23] See, for instance, J.M.Blatt & V.F.Weisskopf, Theoretical Nuclear Physics (New York, John Wiley & Sons, 1952) pp. 789 ff.

$$\mathfrak{u}[\omega, j, \ell, M](x, y, z, t)$$

$$= b[\ell,\omega](r,t)\sum_{m,\mu} C_{\ell 1}(j, M; m, \mu)\, Y_\ell^m(\theta,\varphi)\, \mathfrak{e}[\mu] \qquad (6.15)$$

where b should be determined by (6.10) and can be expressed in terms of Bessel functions of half-odd-integer order, $B_{\ell+\frac{1}{2}}(\omega r)$:

$$b[\ell,\omega](r,t) = (1/\sqrt{r})\, B_{\ell+\frac{1}{2}}(\omega r)\, e^{-i\omega t} \qquad (6.16)$$

which can be normalized by

$$\lim_{R\to\infty} (1/R)\int_0^R (1/\sqrt{r})\, B_{\ell+\frac{1}{2}}(\omega r)\,(1/\sqrt{r})\, B_{\ell+\frac{1}{2}}(\omega' r)\, r^2\, dr = \delta(\omega,\omega') . \qquad (6.17)$$

The left-hand side of (6.17) can be expressed again in terms of Bessel functions[24], and assuming assymptotic expressions of the Bessel functions for large R, we can satisfy (6.17). In particular, since we are interested in the solutions in the empty space, we have to take a superposition of incoming and outgoing waves, which is free from singularity at the origin:

$$B_{\ell+\frac{1}{2}}(\omega r) = (1/\sqrt{\pi\omega})\, J_{\ell+\frac{1}{2}}(\omega r) . \qquad (6.18)$$

Using (6.15), we can expand $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{x},t) = (1/\sqrt{2R})\sum_\omega\sum_j\sum_\ell\sum_M \omega^{-\frac{1}{2}}\left[g(\omega,j,\ell,M)\,\mathfrak{u}[\omega,j,\ell,M](\mathbf{x},t) + \text{herm.conj.}\right] \qquad (6.19)$$

On account of the completeness of the $\mathfrak{u}$'s , we have the inverse orthogonality:

$$\lim_{R\to\infty} \sum_\omega\sum_j\sum_\ell\sum_M (1/R)\, u^{\alpha *}[\omega,j,\ell,M](\mathbf{x},t)\, u^{\beta}[\omega,j,\ell,M](\mathbf{x}',t)$$

$$= \delta(\mathbf{x}-\mathbf{x}')\,\delta(\alpha,\beta) . \qquad (6.20)$$

With the help of (6.20), we can easily show that the commutation rules:

$$[g(\omega,j,\ell,M), \bar{g}(\omega',j',\ell',M')]_- = \delta(\omega,\omega')\,\delta(j,j')\,\delta(\ell',\ell')\,\delta(M,M'), \text{etc.} \qquad (6.21)$$

lead to the ordinary commutation rules of $\mathbf{A}$.

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[24] G.N. Watson, Theory of Bessel Functions (Cambridge University Press, 1922). Eqs (8)(11), p. 134-135.

Next, in order to determine the effect of R and M on the g's, we first have to determine the sign-change of the $\mathfrak{u}$'s for $\mathcal{R}$ and $\mathcal{M}$ applied to their argument ($\mathbf{x}, t$). Using the symbol introduced in (2.73), we get from (6.15)

$$\left.\begin{aligned} \mathcal{R}_{xyz}\, \mathfrak{u}[\omega, j, \ell, M] &= (-1)^{\ell}\, \mathfrak{u}[\omega, j, \ell, M] \\ \mathcal{R}_{t}\, \mathfrak{u}[\omega, j, \ell, M] &= (-1)^{j+\ell+M+1}\, \mathfrak{u}^{*}[\omega, j, \ell, -M] \end{aligned}\right\} \quad (6.22)$$

The first relation represents the well-known transformation rule of Y_{ℓ}^{m} while the second relation can be derived by noticing:

$$\left.\begin{aligned} &C_{\ell 1}(j, -M; -m, -\mu) = (-1)^{j+\ell+1}\, C_{\ell 1}(j, M, m, \mu), \\ &Y_{\ell}^{-m} = (-1)^{m}\, Y_{\ell}^{*m} \qquad , \qquad \mathbb{e}[-\mu] = (-1)^{\mu}\, \mathbb{e}^{*}[\mu], \\ &b[\ell, \omega](r, -t) = b^{*}[\ell, \omega](r, t) \end{aligned}\right\} \quad (6.23)$$

Then, substituting (6.19) in (6.1), we obtain

$$\left.\begin{aligned} C\, g(\omega, j, \ell, M)\, C^{-1} &= -g(\omega, j, \ell, M) \\ (R^{-1}\, g(\omega, j, \ell, M)\, R)^{T} &= (-1)^{j+\ell+M}\, \bar{g}(\omega, j, \ell, -M) \\ M\, g(\omega, j, \ell, M)\, M^{-1} &= (-1)^{\ell+1}\, g(\omega, j, \ell, M) \end{aligned}\right\} \quad (6.24)$$

The effect of C in this representation is essentially the same as in (6.4). The important feature of this representation is that, except for the factor $(-1)^{\ell+1}$, the effect of M is to transform a g into itself with the same (ω, j, ℓ, M). This is of course a consequence of the fact that $\mathcal{M}$ commutes with the quantities defining the eigen-states in this representation.

We can satisfy the third relation of (6.24) by

$$M = \pm \prod_{\substack{\ell \\ \text{even}}} (-1)^{N(\omega, j, \ell, M)} \quad , \qquad (6.25)$$

where the double-sign is not determined even by the condition $M^{2} = 1$. However, if we want to have

$$M\, \Psi_{vac} = +\Psi_{vac} \quad , \qquad (6.26)$$

we have to take the upper sign. Any state in this representation may be expressed by

$$\Psi = \sum_i a_i \, \bar{g}_1 \bar{g}_2 \cdots \Psi_{vac} \; , \tag{6.27}$$

with the g's used in (6.19). If the number of g's with ℓ = even in each term of (6.27) is even (odd), the state (6.27) is an eigen-state of M corresponding to the eigen-value +1 (-1):

$$M\Psi = \pi_s \Psi \quad , \quad \pi_s = \pm 1 \; . \tag{6.28}$$

The value of π_s is the "space-parity" of the state Ψ . The eigen-states (6.28) of M are "self-miraged" states in the sense of (3.54).

In particular, if a single quantum is excited:

$$\Psi = \bar{g} \, \Psi_{vac} \; , \tag{6.29}$$

$(-1)^{\ell+1}$ gives the parity of the state, where ℓ is the orbital angular momentum of $\bar{g}$. Comparing this result with (6.22), we see that the parity of the state (6.29) is the opposite to the parity of the corresponding $\mathfrak{A}$ in the c-number theory. This is, however, in agreement with the usual definition of parity of electromagnetic field, since the parity of $\mathfrak{A}$ is the same as the parity of the electric field and opposite to the parity of the magnetic field, and the parity is defined usually by the magnetic field $\mathbb{H}$:[25]

$$\mathbb{H}(x) = \pm \mathbb{H}(-x) \tag{6.30}$$

The only caution we have to take is that the c-number theoretical definition of parity (6.30) should be re-interpreted in the q-number theory through (6.22) and not through

$$(\Psi, \mathbb{H}(x)\Psi) = \pm (\Psi, \mathbb{H}(-x)\Psi)$$

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[25] J.M.Blatt and V.F.Weisskopf, loc. cit., p.585.

For, Ψ being a self-miraged state, we have always the **positive** sign in this equation. See Tables III and V, Part I.

If we introduce a distinction between the "electric" radiation and "magnetic" radiation by

$$\left.\begin{aligned} &\text{"electric"}: \; j = \ell + 1 \,, \\ &\text{"magnetic"}: \; j = \ell \,, \end{aligned}\right\} \qquad (6.31)$$

we have the rule that the parity of an electric radiation is determined by $(-1)^j$, and that of a magnetic radiation by $(-1)^{j+1}$.

The charge-parity can be defined by

$$\left.\begin{aligned} &C = \Pi\,(-1)^N \,, && C\,\Psi_{vac} = \Psi_{vac} \\ &C\,\Psi = \pi_c\,\Psi \,, && \pi_c = \pm 1 \end{aligned}\right\} \qquad (6.32)$$

The reversion operator can be written

$$\left.\begin{aligned} R &= (-1)^{\Theta} \\ \Theta &= \frac{1}{2}\sum_{\omega}\sum_{j}\sum_{\ell}\sum_{M}{}' \Big[N(\omega,j,\ell,M) + N(\omega,j,\ell,-M) \\ &\qquad - (-1)^{j+\ell+M}\;\bar{g}(\omega,j,\ell,M)\,g(\omega,j,\ell,-M) \\ &\qquad - (-1)^{j+\ell+M}\;\bar{g}(\omega,j,\ell,-M)\,g(\omega,j,\ell,M)\Big] \,. \end{aligned}\right\} \qquad (6.33)$$

C. Mixed Representation

In this representation, linear momentum $\mathbb{p}$ and circular polarization, i.e., spin angular momentum $\mathbb{S}$ in the direction of propagation $\mathbb{p}$ are used. For instance, for the waves propagating in the z-direction, the circular polarization can be expressed by S_z. Obviously S_z does not commute with p_x and p_y, but by taking those eigen-functions for which $p_x = 0$ and $p_y = 0$, we can still use S_z to characterize eigen-functions. Thus, we can write for such waves

$$\mathbb{A}(x,t) = (2V)^{-\frac{1}{2}} \sum \omega^{-\frac{1}{2}} \big[g(k_z,\mu)\,\mathfrak{U}[k_z,\mu](x,t) + \text{herm. conj.} \big] \qquad (6.34)$$

with

$$\mathfrak{u}[k_z,\mu](\mathfrak{x},t) = \exp(ik_z z - i\omega t)\,\mathfrak{e}[\mu]\ ,$$

$$|k_z| = \omega\ , \tag{6.35}$$

where $\mathfrak{e}[0]$, (6.12), should be absent on account of transversality (6.7). From this follows

$$\left.\begin{aligned} C\,g(k_z,\mu)\,C^{-1} &= -g(k_z,\mu)\ , \\ (R^{-1}g(k_z,\mu)\,R)^T &= \bar{g}(-k_z,-\mu)\ , \\ M\,g(k_z,\mu)\,M^{-1} &= -g(-k_z,\mu)\ . \end{aligned}\right\} \tag{6.36}$$

For the purpose of a later application, let us also consider the simple reflection $\mathfrak{M}_x$ of the x-axis: (x, y, z, t) → (-x, y, z, t). The operator M_x corresponding to $\mathfrak{M}_x$ should then be defined by

$$\begin{aligned} M_x A_x(x,y,z,t)M_x^{-1} &= -A_x(-x,y,z,t)\ , \\ M_x A_{y,z}(x,y,z,t)M_x^{-1} &= +A_{y,z}(-x,y,z,t)\ . \end{aligned} \tag{6.37}$$

Since $\mathfrak{e}[1]$ and $\mathfrak{e}[-1]$ differ only in the sign of $\mathfrak{e}_x$, (6.12), we obtain

$$M_x\,g(k_z,\mu)\,M_x^{-1} = g(k_z,-\mu)\ . \tag{6.38}$$

One of the advantages of this mixed representation is that we can easily discuss the connection between the spin and the linear polarization of a photon. The operators, in the sense of O in (6.8), representing the linear polarization are

$$\pi_x = \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}\ ,\quad \pi_y = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}\ ,\quad \pi_z = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix}\ , \tag{6.39}$$

and the corresponding physical quantities Q are

$$\Pi_\alpha = -i\int(\dot{\mathbb{A}}\cdot\pi_\alpha\mathbb{A})\,dV\ ,\quad (\alpha = x,y,z)\ . \tag{6.40}$$

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

We can give more explicit expressions to Π_α by the use of (6.34) and (6.12). In particular, for a given value of k_z, we get

$$\begin{aligned}\Pi_x(k_z) &= \tfrac{1}{2}[N(k_z,+1)+N(k_z,-1)-\bar{g}(k_z,-1)g(k_z,+1)\\ &\quad -\bar{g}(k_z,+1)g(k_z,-1)]\,,\\ \Pi_y(k_z) &= \tfrac{1}{2}[N(k_z,+1)+N(k_z,-1)+\bar{g}(k_z,-1)g(k_z,+1)\\ &\quad +\bar{g}(k_z,+1)g(k_z,-1)]\,.\end{aligned} \tag{6.41}$$

These quantities will be profitably used in our discussion of the positronium decay in #13.

#7 Charged Spinor Field Linear Momentum Representation

In the case of a charged spinor field, we have to consider the free Lagrangean given in (4.5) and the current-density given in (4.6). When these quantities are substituted in (3.11), (3.12) and (3.14), we should have

$$\left.\begin{array}{lll} \rho_C = \rho_R = \rho_M = +1 & \text{for } \mathcal{L}, \\ -\rho_C = -\rho_R = -\rho_M = +1 & \text{for } S^\alpha, \ (\alpha = 1, 2, 3), \\ -\rho_C = \rho_R = \rho_M = +1 & \text{for } S^0. \end{array}\right\} \tag{7.1}$$

These requirements can be satisfied by

$$C\psi C^{-1} = -e^{i\alpha} K\psi^\times , \qquad C\psi^\times C^{-1} = e^{-i\alpha}\psi K^{-1} , \tag{7.2}$$

$$(R^{-1}\psi(t)R)^T = e^{i\beta} E_0 K \psi^\times(-t), \qquad (R^{-1}\psi^\times(t)R)^T = e^{-i\beta}\psi(-t)K^{-1}E_0 , \tag{7.3}$$

$$M\psi(\mathbf{x})M^{-1} = e^{i\gamma}E_{123}\psi(-\mathbf{x}), \qquad M\psi^\times(\mathbf{x})M^{-1} = -e^{-i\gamma}\psi^\times(-\mathbf{x})E_{321} . \tag{7.4}$$

It can easily be seen, with the help of (2.18) and (2.20), that these transformations are compatible with the unitarity of C, R and M. The usual commutation rules of ψ are left unchanged by these transformations. The combination of the two relations of (7.2) gives $(C)^2\psi(C^{-1})^2 = \psi$, agreeing with $C^2 = 1$ irrespectively of α. The reiteration of the transformations (7.3) gives

$$R^T R^{-1}\psi(t) R R^{T-1} = -\psi(t),$$

implying

$$R^T R^{-1} = \Delta . \tag{7.5}$$

We shall see in Appendix a method by which $R^T R^{-1}$ can be made a c-number using an 8-component spinor formalism.

The double mirage, according to (7.4), yields

$$M^2 \psi (M^{-1})^2 = - e^{2i\gamma} \psi , \tag{7.6}$$

which means, among other possiblities[26], that

$$\left.\begin{aligned} M^2 &= 1 \quad \text{for} \quad e^{i\gamma} = \pm i , \\ M^2 &= \Delta \quad \text{for} \quad e^{i\gamma} = \pm 1 . \end{aligned}\right\} \tag{7.7}$$

According to (3.50), we shall adopt the former alternative.

We can easily see that the double-time-reversal in the sense of (3.56) can be a c-number or Δ , or further a more general W. If we first apply (7.2) and then (7.3) we get

$$(R^{-1} C \psi(t) C^{-1} R)^T = - e^{i(\alpha - \beta)} E_0 \psi(-t) . \tag{7.8}$$

If we invert the order of these two, we get

$$(C^{-1T} R^{-1} \psi(t) R C^T)^T = - e^{-i(\alpha-\beta)} E_0 \psi(-t) . \tag{7.9}$$

By putting (3.57), we obtain from (7.8) and (7.9)

$$W^{-1} \psi W = e^{2i(\alpha-\beta)} \psi , \tag{7.10}$$

which means that if $\alpha = \beta$ or $\alpha = \beta \pm \pi$ then $W =$ c-number, and that if $\alpha = \beta \pm \pi/2$ then $W = \Delta$. In the latter case, the double-time-reversal in the sense of (3.56) will become a c-number on account of (3.58). The more general case of W where α and β are arbitrary will be included in our discussion of the superselection rules.

The relation (7.2) satisfies the requirements (7.1) for (4.5) and (4.6) only on the assumption that ψ and $\psi^\times$ anticommute. As a result of the application of the commutation relations, there appears a c-number additional

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[26] See footnote 16.

term. In order to avoid such an additional term, it is desirable to write $\mathcal{L}$ and S^{μ} in the form

$$\left.\begin{aligned} \mathcal{L} \to \tfrac{1}{2}(\mathcal{L} + \rho_c C \mathcal{L} C^{-1}) &= -\tfrac{1}{4}\psi^{\times}[E_{\mu}(\partial/\partial x_{\mu}) - E_{\mu}(\overleftarrow{\partial/\partial x_{\mu}}) + 2imE_5]\psi \\ &\quad - \tfrac{1}{4}\psi[E_{\mu}^{T}(\partial/\partial x_{\mu}) - E_{\mu}^{T}(\overleftarrow{\partial/\partial x_{\mu}}) - 2imE_5^{T}]\psi^{\times}, \\ S^{\mu} \to \tfrac{1}{2}(S^{\mu} + \rho_c C S^{\mu} C^{-1}) &= \tfrac{i}{2}(\psi^{\times}E^{\mu}\psi - \psi E^{\mu T}\psi^{\times}) . \end{aligned}\right\} \quad (7.11)$$

Naturally, (7.11) is written so that the requirements (7.1) are automatically satisfied, but if we assume the Bose-statistics, (7.11) becomes essentially a c-number.

We shall now proceed to determine the effect of C, R and M in the linear momentum representation. In this representation, we use the following mutually commuting hermitian operators:

$$\left.\begin{aligned} &p_{\alpha} = -i(\partial/\partial x_{\alpha}) \;,\; (\alpha = 1,2,3) \;, && \vartheta = i(\partial/\partial t) \;, \\ &H_0 = E_0 E_{\alpha} p_{\alpha} + E_0 E_5 m \;, && \sigma = E_0 E_5 E_{\alpha} p_{\alpha}/2p \;, \end{aligned}\right\} \quad (7.12)$$

with

$$p = \left|\left(\sum_{\alpha} p_{\alpha}^2\right)^{\frac{1}{2}}\right| \quad (7.13)$$

σ means the spin in the direction of propagation p_{α} . Denoting the eigenvalues of p_{α} , $H_0 = \vartheta$, σ respectively by $\mathbb{k}$, ω, μ, we use the expression:

$$\psi[\mathbb{k}, \omega, \mu](\mathbb{x}, t) \quad , \quad \omega = \pm (\mathbb{k}^2 + m^2)^{\frac{1}{2}} \quad , \quad (7.14)$$

for the corresponding eigen-functions. The absolute value of ω is determined by $\mathbb{k}$, but its sign is an independent quantum number. Thus we can expand any solution of the wave-equation as

$$\psi(\mathbb{x}, t) = (V)^{-\frac{1}{2}} \sum_{\lambda} a_{\lambda} \psi[\lambda](\mathbb{x}, t) \quad (7.15)$$

where λ stands collectively for ($\mathbb{k}$, ω , μ).

Next we investigate the effect of the transformations appearing on the right sides of the relations (7.2) (7.3) (7.4) on the eigen-functions (7.14). In other words, we shall study the behaviors of the functions:

$$K\psi^{\times}[\lambda](x,t)\,, \qquad E_0K\psi^{\times}[\lambda](x,-t)\,, \qquad E_{123}\psi[\lambda](-x,t)\,, \tag{7.16}$$

towards the operators (7.12), as compared with the behavior of the original function (7.14).

The transformations from (7.14) to (7.16) can be decomposed into: multiplication by K, hermitian conjugation, multiplication by J (to obtain $\psi^{\times}$), and multiplication by

$$T \equiv iE_0\Omega_t \qquad \text{and} \qquad P \equiv E_{05}\,\Omega_{xyz} \qquad (E_{05} = -iE_{123}) \tag{7.17}$$

where the meaning of Ω -symbol is given by (2.73). The factor i is inserted to make T and P hermitian, but does not have much importance.

The defining equations for (7.14) have the form

$$O\psi = O'\psi \tag{7.18}$$

where O stands for any one of the operators in (7.12) and O' stands for the eigen-value. From (7.18) follows, by the application of the above-mentioned transformations,

$$\left.\begin{aligned} (K(J^{-1}OJ)^{T}K^{-1})\;K\psi^{\times}(x,t) &= O'\,K\,\psi^{\times}(x,t) \\ (T\,K(J^{-1}OJ)^{T}K^{-1}T)\;E_0K\psi^{\times}(x,-t) &= O'\,E_0K\psi^{\times}(x,-t) \\ (POP^{-1})\;E_{123}\,\psi(-x,t) &= O'\,E_{123}\psi(-x,t) \end{aligned}\right\} \tag{7.19}$$

showing that the functions mentioned in (7.16) have the eigen-values O' for the operators appearing on the left sides of (7.19), instead of the operator O for which $\psi(x,t)$ has the eigen-value O'. To characterize these operators, we introduce the following sign-functions:

$$\left.\begin{aligned} J^{-1} O J &= \rho_J \bar{O} = \rho_J O \\ K^{-1} O K &= \rho_K O^T \\ T^{-1} O T &= \rho_T O \\ P^{-1} O P &= \rho_P O \end{aligned}\right\} \qquad (7.20)$$

where O^T should be understood as the transpose of O, as explained at the end of Subsection #2.D. With the help of these ρ's, we can write (7.19) more compactly:

$$\left.\begin{aligned} O\,\psi(x,t) &= O'\,\psi(x,t) \\ O\,K\,\psi^x(x,t) &= \rho_J\rho_K\,O'\,K\,\psi^x(x,t) \\ O\,E_0 K\,\psi^x(x,-t) &= \rho_J\rho_K\rho_T\,O'\,E_0 K\,\psi^x(x,-t) \\ O\,E_{123}\,\psi(-x,t) &= \rho_P\,O'\,E_{123}\,\psi(-x,t) \end{aligned}\right\} \qquad (7.21)$$

The ρ's for the operators mentioned in (7.12) are listed in Table I.

	ρ_J	ρ_K	ρ_T	ρ_P	$\rho_J\rho_K$	$\rho_J\rho_K\rho_T$
P_α	+	−	+	−	−	−
$\mathcal{J}$	+	−	−	+	−	+
H_0	−	+	−	+	−	+
σ	+	+	+	−	+	+

Table I
The sign functions ρ's defined in (7.20) and used in (7.21) for various operators introduced in (7.12).

Reading the column for $\rho_J\rho_K$ in Table I, we see on account of (7.21), that $K\,\psi^x(x,t)$ has the opposite signs for momentum, energy and the same sign for spin σ as compared with $\psi(x,t)$. Since σ is the spin in the direction

of $\mathbf{k}$, the same sign of σ here actually means the opposite spin. In any event, if $\psi[\lambda]$ represents a positive energy, $K\psi^{\times}[\lambda]$ will represent a negative energy. Therefore, we can write instead of (7.15)

$$\psi(\mathbf{x},t) = V^{-\frac{1}{2}} \sum_{\lambda} \left[a_{\lambda}\psi[\lambda](\mathbf{x},t) + \bar{b}_{\lambda}\psi^{\times}[\lambda](\mathbf{x},t)K \right] . \tag{7.22}$$

An important difference between (7.15) and (7.22) is that λ in (7.22) stands only for ($\mathbf{k}$, μ) while λ in (7.21) stood for (k , ω , μ). Hereafter $\omega = +|(k^2+m^2)^{\frac{1}{2}}|$. The spinor adjoint of (7.22) is, on account of $\overline{(\psi^{\times}K)}J^{-1} = \psi K^{-1}$,

$$\psi^{\times}(\mathbf{x},t) = V^{-\frac{1}{2}} \sum_{\lambda} \left[\bar{a}_{\lambda}\psi^{\times}[\lambda](\mathbf{x},t) + b_{\lambda}\psi[\lambda](\mathbf{x},t)K^{-1} \right] . \tag{7.23}$$

The fact that $K\psi^{\times}(=-\psi^{\times}K)$ has the opposite values of $\mathbf{p}$ and spin is just compensated in the q-number theory by the fact that, in an expression like $\psi^{\times}JO\psi = \bar{\psi}O\psi$, the amplitudes a_{λ} and $\bar{a}_{\lambda}$ stand in the order $\bar{a}_{\lambda}a_{\lambda}$ while the amplitudes b_{λ} and $\bar{b}_{\lambda}$ stand in the order $b_{\lambda}\bar{b}_{\lambda} = 1 - \bar{b}_{\lambda} b_{\lambda}$. Therefore, $\bar{b}_{\lambda} b_{\lambda}$ will represent the number of particles having the same values of λ as $\bar{a}_{\lambda} a_{\lambda}$, only with opposite charge.

Next reading the column for $\rho_J\rho_K\rho_T$, we see that $E_0K\psi^{\times}(\mathbf{x},-t)$ has the opposite $\mathbf{k}$ and the same value of ϑ , H and σ as $\psi(\mathbf{x},t)$. Since $\mathbf{k}$ is opposite, the same value of σ actually means the flipping of spin. Therefore we can write

$$E_0K\psi^{\times}[\mathbf{k},\mu](\mathbf{x},t) = e^{i\alpha}\psi[-\mathbf{k},\mu](\mathbf{x},-t) \tag{7.24}$$

where μ on the left side refers to $\mathbf{k}$ and μ on the right refers to $-\mathbf{k}$. If we solve for $\psi[-\mathbf{k},\mu]$, keeping the same values of μ , $\mathbf{k}$ $\mathbf{x}$ and t, we obtain by the use of (2.20)

$$E_0K\psi^{\times}[-\mathbf{k},\mu](\mathbf{x},-t) = -e^{i\alpha}\psi[\mathbf{k},\mu](\mathbf{x},t) \tag{7.25}$$

If we re-name $-\mathbb{k}$ and $-t$ as $\mathbb{k}$ and t, (7.24) and (7.25) become contradictory. This contradiction is of course only apparent, because the same value of μ referring to $-\mathbb{k}$ and to $\mathbb{k}$ means opposite spins. The simplest way to avoid the confusion would be to divide the space into two hemispaces $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathbb{x}$ and $-\chi$ cannot belong to the same hemi-space, and to intorduce a new spin-index τ such that

$$\left.\begin{array}{ll} \tau = +1 & \text{if the actual spin direction is in } \mathcal{V}_1, \\ \tau = -1 & \text{if the actual spin direction is in } \mathcal{V}_2. \end{array}\right\} \qquad (7.26)$$

Then we can write consistently

$$E_0 K \psi^\times[\mathbb{k},\tau](\mathbb{x},t) = \tau\,\psi[-\mathbb{k},-\tau](\mathbb{x},-t) \qquad (7.27)$$

suitably choosing $e^{i\alpha}$. Writing this way, we can now freely change the sign of $\mathbb{k}$ and t. (7.26) is just a matter of convention, and we still consistently use the relativistic eigen-functions.

Next, according to the column for ρ_P in Table I, we see that $E_{123}\psi(-\chi, t)$ belongs to the same energy and the opposite $\mathbb{p}$ and μ (i.e., the same spin). Therefore we can write

$$i E_{123}\,\psi[\mathbb{k},\tau](\mathbb{x},t) = \psi[-\mathbb{k},\tau](-\mathbb{x},t) \qquad (7.28)$$

The factor i is added to secure the symmetry with regard to the signs of $\mathbb{k}$ and $\mathbb{x}$. It can easily be shown that there is no conflict between (7.27) and (7.28), in the sense that the application of the transformation (7.27) first and (7.28) next gives the same result as the application of (7.28) first and (7.27) next.

Finally using the explicit expressions of ψ as plane waves:

$$\psi[\mathbb{k},\tau] = \alpha[\mathbb{k},\tau]\exp(+i\mathbb{k}\mathbb{x}, -i\omega t), \qquad (7.29)$$

we can re-write (7.22) and (7.23) as

$$\left.\begin{aligned}\psi(\mathbf{x},t) &= V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}\big[g_+(\mathbf{k},\tau)\alpha[\mathbf{k},\tau]\exp(+-)\\ &\qquad + \bar{g}_-(\mathbf{k},\tau)\alpha^{x}[\mathbf{k},\tau]K\exp(-+)\big],\\ \psi^{x}(\mathbf{x},t) &= V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}\big[\bar{g}_+(\mathbf{k},\tau)\alpha^{x}[\mathbf{k},\tau]\exp(-+)\\ &\qquad + g_-(\mathbf{k},\tau)\alpha[\mathbf{k},\tau]K^{-1}\exp(+-)\big].\end{aligned}\right\} \qquad (7.30)$$

Using (7.27), we can also write, for instance, for ψ^{x}

$$\begin{aligned}\psi^{x}(\mathbf{x},t) = V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}(-\tau)\big[&\bar{g}_+(\mathbf{k},\tau)\alpha(-\mathbf{k},-\tau)K^{-1}E_0\exp(-+)\\ &+ g_-(\mathbf{k},\tau)\alpha^{x}(-\mathbf{k},-\tau)E_0\exp(+-)\big]\,.\end{aligned} \qquad (7.31)$$

and using (7.28), we can write, for instance, for ψ

$$\begin{aligned}\psi(\mathbf{x},t) = V^{-\frac{1}{2}}\sum_{\mathbf{k}}\sum_{\tau}\big[&\bar{g}_+(\mathbf{k},\tau)(iE_{123})\alpha(-\mathbf{k},\tau)\exp(+-)\\ &+ \bar{g}_-(\mathbf{k},\tau)\alpha^{x}(-\mathbf{k},\tau)(iE_{321})K\exp(-+)\big].\end{aligned} \qquad (7.32)$$

Substituting first (7.30) into (7.2), we get

$$\left.\begin{aligned}C\,g_+(\mathbf{k},\tau)C^{-1} &= e^{i\alpha}g_-(\mathbf{k},\tau)\,,\\ C\,g_-(\mathbf{k},\tau)C^{-1} &= e^{-i\alpha}g_+(\mathbf{k},\tau)\,.\end{aligned}\right\} \qquad (7.33)$$

Substituting (7.30) and (7.31) into (7.3), we obtain

$$\left.\begin{aligned}(R^{-1}g_+(\mathbf{k},\tau)R)^{T} &= -e^{i\beta}\tau\,\bar{g}_+(-\mathbf{k},-\tau)\,,\\ (R^{-1}g_-(\mathbf{k},\tau)R)^{T} &= -e^{-i\beta}\tau\,\bar{g}_-(-\mathbf{k},-\tau)\,.\end{aligned}\right\} \qquad (7.34)$$

Substituting (7.30) and (7.32) into (7.4), we obtain

$$\left.\begin{aligned}M\,g_+(\mathbf{k},\tau)M^{-1} &= -ie^{i\gamma}g_+(-\mathbf{k},\tau)\,,\\ M\,g_-(\mathbf{k},\tau)M^{-1} &= -ie^{-i\gamma}g_-(-\mathbf{k},\tau)\,.\end{aligned}\right\} \qquad (7.35)$$

which should lead to $M^2 = 1$ for $\gamma = \pm\pi/2$.

The charge-conjugation operator $\mathbb{C}$ as defined by (7.33) can easily be obtained, namely : $C = C^{-1} = \bar{C} = \prod_{\mathbf{k}}\prod_{\tau} C(\mathbf{k},\tau)$

$$C(\mathbf{k},\tau) = 1 - \underset{\sim}{N}_+(\mathbf{k},\tau) - \underset{\sim}{N}_-(\mathbf{k},\tau) + e^{i\alpha}\bar{g}_+(\mathbf{k},\tau)g_-(\mathbf{k},\tau) + e^{-i\alpha}\bar{g}_-(\mathbf{k},\tau)g_+(\mathbf{k},\tau). \qquad (7.36)$$

Next for the reversion operator R, we have to satisfy

$$\left.\begin{aligned} R^{-1} g_\pm(\mathbb{k},+) R &= -e^{\pm i\beta} g_\pm(-\mathbb{k},-) \\ R^{-1} g_\pm(\mathbb{k},-) R &= +e^{\pm i\beta} g_\pm(-\mathbb{k},+) \end{aligned}\right\} \tag{7.37}$$

where the usual representation of g and $\bar{g}$ is assumed in which g and $\bar{g}$ have real matrix elements. We can first get rid of numerical factors by putting $R = R_0 R'$,

$$R_0 = \prod_\pm \prod_{\mathbb{k}} (-e^{\pm i\beta})^{N_\pm(\mathbb{k},+)} (+e^{\pm i\beta})^{N_\pm(\mathbb{k},-)} \quad . \tag{7.38}$$

Then (7.37) becomes

$$R'^{-1} g_\pm(\mathbb{k},\tau) R' = g_\pm(-\mathbb{k},-\tau) \tag{7.39}$$

which is obviously satisfied by

$$R' = \prod_{\mathbb{k}}{}' \prod_\pm R'_\pm(\mathbb{k})$$

$$R'_\pm(\mathbb{k}) = 1 - N_\pm(\mathbb{k},+) - N_\pm(-\mathbb{k},-) + \bar{g}_\pm(\mathbb{k},+) g_\pm(-\mathbb{k},-) + \bar{g}_\pm(-\mathbb{k},-) g_\pm(\mathbb{k},+) \tag{7.40}$$

where the multiplication symbol $\prod'$ with a prime means that $\mathbb{k}$ should be taken from one hemi-space. Now we can confirm that the R thus determined in fact satisfies $R^{-1} R^T = \Delta$. Noticing that $R_0^T = R_0$ and $R'^T = R'$ we get

$$R^{-1} R^T = R'^{-1} R_0^{-1} R' R_0$$

From (7.39), we see that the effect of the transformation by R' is such that $N_\pm(\mathbb{k},+)$ and $N_\pm(-\mathbb{k},-)$ are interchanged. This amounts to an interchange of the factor of the type $(-e^{\pm i\beta})^N$ and the factor of the type $(+e^{\pm i\beta})^N$ in the expression of R_0 or of R_0^{-1} , which results in a multiplication by $(-1)^N$ for all oscillators. Thus we have

$$R'^{-1} R_0^{-1} R' = \Delta R_0^{-1} ,$$

and

$$R^{-1} R^T = \Delta \; . \tag{7.41}$$

Finally the mirage operator M satisfying (7.35) can be written as

$$M = M' M_0 , \tag{7.42}$$

with

$$M_0 = \prod_{\mathbf{k}} \prod_{\tau} (+ i e^{-i\gamma})^{N_+(\mathbf{k},\tau)} (+ i e^{+i\gamma})^{N_-(\mathbf{k},\tau)} . \tag{7.43}$$

and

$$\left. \begin{aligned} M' = \prod_{\mathbf{k}}{}' \prod_{\tau} \prod_{\pm} \big[1 - N_\pm(\mathbf{k},\tau) - N_\pm(-\mathbf{k},\tau) \\ + \bar{g}_\pm(\mathbf{k},\tau) g_\pm(-\mathbf{k},\tau) + \bar{g}_\pm(-\mathbf{k},\tau) g_\pm(\mathbf{k},\tau) \big] . \end{aligned} \right\} \tag{7.44}$$

If we make the square of M, we get

$$M^2 = M_0^2 = \prod_{\mathbf{k}} \prod_{\tau} (-e^{-2i\gamma})^{N_+(\mathbf{k},\tau)} (-e^{+2i\gamma})^{N_-(\mathbf{k},\tau)} , \tag{7.45}$$

which is unity for $\gamma = \pm\pi/2$, and Δ for $\gamma = 0$ or $\pm\pi$, in agreement with (7.7). To satisfy $M^2 = 1$, we should take

$$M_0 = \prod_{\mathbf{k}} \prod_{\tau} (-1)^{N_+(\mathbf{k},\tau)} \qquad \text{or} \qquad M_0 = \prod_{\mathbf{k}} \prod_{\tau} (-1)^{N_-(\mathbf{k},\tau)} . \tag{7.46}$$

The mirage operator thus defined satisfies

$$M \Psi_{vac} = \Psi_{vac} . \tag{7.47}$$

Evidently this is a matter of convention for we could as well take the negative of (7.42) as M.

#8. Charged Spinor Field - Angular Momentum Representation

In this representation, the operators used to characterize eigen-states are, besides H and ϑ (7.12), the following three operators which commute among themselves and with H and ϑ :

$$I = L_\alpha E_\alpha + E_0 E_5 \,, \quad (\alpha = 1, 2, 3), \tag{8.1}$$

$$J_3 = L_3 + \tfrac{1}{2} E_0 E_5 E_3 \,, \tag{8.2}$$

$$P = E_0 E_5 \Omega_{\mathbf{x}} \,, \tag{8.3}$$

where

$$L_1 = -i\left[x_2(\partial/\partial x_3) - x_3(\partial/\partial x_2)\right], \text{ etc.}, \tag{8.4}$$

and

$$\Omega_{\mathbf{x}} f(\mathbf{x}) = f(-\mathbf{x}) \,, \tag{8.5}$$

or

$$(\mathbf{x}' | \Omega_{\mathbf{x}} | \mathbf{x}'') = \delta(\mathbf{x}' + \mathbf{x}'') \,. \tag{8.6}$$

It is convenient, in this representation, to introduce the following operators, X, Y :

$$X = (i/r)\, E_\alpha x_\alpha \Omega_{\mathbf{x}} \qquad (\alpha = 1, 2, 3) \tag{8.7}$$

$$Y = (i/r)\,(E_{12} x_3 + E_{23} x_1 + E_{31} x_2) \tag{8.8}$$

It is easy to see that X , Y and P satisfy the same commutation rules as the spin matrices:

$$\left.\begin{aligned} &X^2 = Y^2 = P^2 = 1 \\ &XY = -YX = iP, \quad YP = -PY = iX, \quad PX = -XP = iY. \end{aligned}\right\} \tag{8.9}$$

These operators, X , Y and P, are analogues of s_1, s_2 and s_3, (2.58) (2.59) and (2.60)

Now noticing that

$$p_1 = -i(\partial/\partial x_1) = (x_1/r)(p_r + i/r) - (x_2/r^2)L_3 + (x_3/r^2)L_2 , \quad \text{etc.,} \tag{8.10}$$

with

$$p_r = -i\left[(\partial/\partial r) + (1/r)\right] = (1/r)(x_1 p_1 + x_2 p_2 + x_3 p_3 - i) \tag{8.11}$$

we can re-write the Hamiltonian by a straightforward substitution in the form,[27]

$$H = -iE_0 X \Omega_{\mathfrak{x}} p_r - iE_0 Y (1/r) I + m P \Omega_{\mathfrak{x}} \tag{8.12}$$

The eigen-functions in this representation will be characterized by ω, $\varkappa$, m and ϱ, which are respectively eigen-values of $H = \vartheta$, I, J_3 and P:

$$\psi[\omega, \varkappa, m, \varrho](\mathfrak{x}, t) . \tag{8.13}$$

It is obvious that if ω is an eigen-value then $-\omega$ is also an eigenvalue. However, in contrast to the case of the linear momentum representation, $|\omega|$ is not determined by the other quantum numbers. Thus, similarly to the case of the linear momentum representation, here too, we have four quantum numbers.

To determine the relationship among eigenfunctions, we calculate the sign-functions ϱ_J, ϱ_K, ϱ_T and ϱ_P, defined by (7.20), for the various operators used in this representation. The results are listed in Table II.

	ϱ_J	ϱ_K	ϱ_T	ϱ_P	$\varrho_J \varrho_K$	$\varrho_J \varrho_K \varrho_T$	ϱ_{PX}
H	−	+	−	+	−	+	+
ϑ	+	−	−	+	−	+	+
P	−	+	−	+	−	+	+
I	−	+	−	+	−	+	+
J_3	+	−	+	+	−	−	−

Table II. The sign-functions ϱ's defined in (7.20) and (8.32) for various operators used in the angular momentum representation.

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[27] P.A.M. Dirac, The Principles of Quantum Mechanics (Oxford Univ. Press, London, 1947) 3rd Edition, p. 268.

We first notice that $\varrho_J \varrho_K = -1$ for all the operators, which means, in virtue of (7.21), that $K\psi^{\times}(x,t)$, as compared with $\psi(x,t)$, has the opposite sign for all the quantities involved, including energy. Therefore we can expand $\psi(x,t)$ with the help of (8.13) and

$$K\psi^{\times}[\omega,\kappa,m,\varrho](x,t)\,, \tag{8.14}$$

while limiting the values of ω to positive values. Thus

$$\psi(x,t) = (1/\sqrt{R})\sum_{\lambda}\left[g_{+}(\lambda)\psi[\lambda](x,t) + \bar{g}_{-}(\lambda)\psi^{\times}[\lambda](x,t)K\right], \tag{8.15}$$

where λ stands for $[\omega,\kappa,m,\varrho]$. As before, the fact that (8.14) has the opposite values of λ is just compensated by the fact that the emission operator $\bar{g}$, instead of the absorption operator g, is standing as its coefficient, where g and $\bar{g}$ anticommute. The commutation rules here should read

$$[g_{\pm}(\omega,\kappa,m,\varrho),\bar{g}_{\pm}(\omega',\kappa',m',\varrho')]_{+} = \delta(\omega,\omega')\delta(\kappa,\kappa')\delta(m,m')\delta(\varrho,\varrho')\delta(\pm,\pm)\,, \text{etc.} \tag{8.16}$$

Next, we see from Table II that $\varrho_J\varrho_K\varrho_T$ is negative only for J_3. This means in the light of (7.21) that $E_0K\psi^{\times}[\lambda](-t)$, as compared with $\psi[\lambda](t)$, has the opposite value of J_3 and the same values of $H = \vartheta$, I and P. Therefore we can put

$$E_0K\psi^{\times}[\omega,\kappa,m,\varrho](x,-t) = e^{i\alpha}\psi[\omega,\kappa,-m,\varrho](x,t) \tag{8.17}$$

Solving for $\psi[-m]$, and taking the spinor adjoint by the help of (2.20), we obtain

$$E_0K\psi^{\times}[\omega,\kappa,-m,\varrho](x,t) = -e^{i\alpha}\psi[\omega,\kappa,m,\varrho](x,-t) \tag{8.18}$$

The apparent dissymmetry between (8.17) and (8.18) can be overcome by noticing that α can depend on m. Thus we can adjust α so that both (8.17) and (8.18) are incorporated in

$$E_0 K \psi[\omega, \varkappa, m, \rho](\mathfrak{x}, -t) = \frac{m}{|m|} \psi[\omega, \varkappa, -m, \rho](\mathfrak{x}, t). \tag{8.19}$$

One of the advantageous features of the angular momentum representation is that P commutes with the other operators defining eigenfunctions. This fact is reflected in Table II by ρ_P being positive for all the operators. Thus we should have, according to (7.21),

$$E_{123} \psi[\omega, \varkappa, m, \rho](-\mathfrak{x}, t) = e^{i\alpha} \psi[\omega, \varkappa, m, \rho](\mathfrak{x}, t),$$

or more directly, because of the fact that $\psi[\lambda]$ is an eigenfunction of P,

$$E_{05} \psi[\omega, \varkappa, m, \rho](-\mathfrak{x}, t) = \rho\, \psi[\omega, \varkappa, m, \rho](\mathfrak{x}, t). \tag{8.20}$$

Now we can determine the effect of C, R and M on the q-number amplitudes g's. Applying (7.2) to (8.15), we obtain

$$C g_\pm(\omega, \varkappa, m, \rho) C^\dagger = e^{\pm i\alpha} g_\mp(\omega, \varkappa, m, \rho), \tag{8.21}$$

which is satisfied by

$$\begin{aligned} C &= C^\dagger = \bar{C} \\ &= \prod_\lambda \left[1 - N_+(\lambda) - N_-(\lambda) + e^{i\alpha} \bar{g}_+(\lambda) g_-(\lambda) + e^{-i\alpha} \bar{g}_-(\lambda) g_+(\lambda)\right]. \end{aligned} \tag{8.22}$$

Application of (7.3) to (8.15) yields, in virtue of (8.19),

$$\left(R^\dagger g_\pm(\omega, \varkappa, m, \rho) R\right)^T = -\frac{m}{|m|} e^{\pm i\beta} \bar{g}_\pm(\omega, \varkappa, -m, \rho), \tag{8.23}$$

which is satisfied by

$$\begin{aligned} R &= R_0 R', \\ R_0 &= \prod_\pm \prod_{\lambda'} \prod_m \left(-\frac{m}{|m|} e^{\pm i\beta}\right)^{N_\pm(\lambda', m)}, \\ R' &= \prod_\pm \prod_{\lambda'} \prod_{|m|} [1 - N_\pm(\lambda', m) - N_\pm(\lambda', -m) + \bar{g}_\pm(\lambda', m) g_\pm(\lambda', -m) + \bar{g}_\pm(\lambda', -m) g(\lambda', m), \end{aligned} \tag{8.24}$$

where λ' stands for $(\omega, \varkappa, \rho)$. This R obviously satisfies

$$R^T R^\dagger = \Delta. \tag{8.25}$$

Finally, the effect of M on the g's is obtained by applying (7.4) to (8.15) with the help of (8.20).

$$\left.\begin{aligned} M\, g_+(\chi',\rho)\, M^{-1} &= -i e^{i\gamma} \rho\, g_+(\chi',\rho) \\ M\, g_-(\chi',\rho)\, M^{-1} &= -i e^{-i\gamma} \rho\, g_-(\chi',\rho) \end{aligned}\right\} \qquad (8.26)$$

which are satisfied by

$$M = \prod_{\chi'} \prod_{\rho} \left(+i\rho e^{-i\gamma}\right)^{N_+(\chi',\rho)} \left(+i\rho e^{+i\gamma}\right)^{N_-(\chi',\rho)} \qquad (8.27)$$

where χ' stands for $(\omega, \varkappa, m)$. We can satisfy the condition $M^2 = 1$, by either one of the following two alternatives:

$$M_1 = \prod_{\chi'} \prod_{\rho} \rho^{N_+(\chi',\rho)} (-\rho)^{N_-(\chi',\rho)}, \qquad (8.28)$$

or

$$M_2 = \prod_{\chi'} \prod_{\rho} (-\rho)^{N_+(\chi',\rho)} \rho^{N_-(\chi',\rho)}.$$

This ambiguity is not surprising since the determination of ρ itself is a matter of convention, we could as well use -P for P. In spite of this, (8.28) can give a definite value of parity when the total number of spinor particles is even. Moreover, we can speak of a definite "relative" parity of two states, each having an odd number of spinor particles, since the "relative" parity (i.e., equality or inequality of parity) has a definite meaning no matter whether we use M_1 or M_2. Both M_1 and M_2 satisfy $\underline{M}\, \Psi_{vac} = \Psi_{vac}$.

For the purpose of a later application, let us briefly consider the reflection with respect to the x-axis: (x, y, z, t) → (-x, y, z, t). The unitary operator M_x is obviously defined by

$$M_x \psi(x,y,z,t)\, M_x^{-1} = e^{i\alpha} E_1 \psi(-x,y,z,t), \qquad (8.29)$$

whose adjoint is, on account of (2.18),

$$M_x \psi^{\times}(x,y,z,t)\, M_x^{-1} = -e^{i\alpha} \psi^{\times}(-x,y,z,t)\, E_1, \qquad (8.30)$$

where α has to be = 0 or $\pm\pi$ in order that $M_x^2 = 1$. The right-hand side of (8.29) is obtained from $\psi(x, y, z, t)$ by applying the operator

$$P_x = E_i \Omega_x \quad . \tag{8.31}$$

The comparison of an eigenfunction $\psi[\lambda]$ and $P_x \psi[\lambda]$ can easily be done by examing

$$P_x^{-1} O P_x = \rho_{Px} O , \tag{8.32}$$

for various operators. The results are listed in Table II, which shows that the only quantity that changes its sign is J_3 . Therefore we can write

$$E_1 \psi[\omega, \kappa, m, \rho](-x, y, z, t) = e^{i\alpha} \psi[\omega, \kappa, -m, \rho](x, y, z, t) \tag{8.33}$$

where $e^{i\alpha}$ can be put = 1 without loss of generality. Applying (8.29) on (8.15), we obtain by the help of (8.33),

$$M_x g_\pm[\lambda', m] M_x^{-1} = e^{i\alpha} g_\pm[\lambda', -m] \tag{8.34}$$

where λ' stands for (ω, κ, ρ) and $e^{i\alpha}$ must be ± 1, in order that $M_x^2 = 1$. M_x can be given an explicit expression:

$$\begin{aligned} M_x = \prod_{\pm} \prod_{\lambda'} \prod_{|m|} \Big[& 1 - N_\pm(\lambda', m) - N_\pm(\lambda', -m) \\ & + (\pm)' \bar{g}_\pm(\lambda', m) g_\pm(\lambda', -m) + (\pm)' \bar{g}_\pm(\lambda', -m) g_\pm(\lambda', m) \Big] \end{aligned} \tag{8.35}$$

where $(\pm)'$ is an arbitrary double sign independent of the double sign of the charge. This M_x satisfies $M_x \Psi_{vac} = \Psi_{vac}$.

#9. Tensorial Quantities Built with Spinors

We have established that charge-conjugation, reversion and mirage with regard to spinor fields are engendered by C, R and M defined by

$$\left.\begin{aligned} C\,\psi(\mathbf{x},t)\,C^{-1} &= e^{i\alpha}\,\psi^{\times}(\mathbf{x},t)\,K\,, \\ (R^{-1}\,\psi(\mathbf{x},t)\,R)^{T} &= e^{i\beta}\,\psi^{\times}(\mathbf{x},-t)\,E_0\,K\,, \\ M\,\psi(\mathbf{x},t)\,M^{-1} &= e^{i\gamma}\,E_{123}\,\psi(-\mathbf{x},t)\quad ,\quad \gamma=\pm\pi/4\,, \end{aligned}\right\} \qquad (9.1)$$

and their adjoint equations. We have also shown that such operators C, R and M actually exist. These relations have been derived from the requirement that the free Lagrangean and the current have the correct transformation for charge-conjugation $\mathcal{C}$, reversion $\mathcal{R}$ and mirage $\mathcal{M}$. Now, we can conversely use these relations (9.1) to find out the transformation rules of various tensorial quantities Q(x, t) built with spinors for $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{M}$. Their transformation properties are determined by

$$\left.\begin{aligned} \rho_C &= (C\,Q(\mathbf{x},t)\,C^{-1})/Q(\mathbf{x},t) \\ \rho_R &= (R^{-1}\,Q(\mathbf{x},t)\,R)^{T}/\,Q(\mathbf{x},-t) \\ \rho_M &= (M\,Q(\mathbf{x},t)\,M^{-1})/\,Q(\mathbf{x},t) \end{aligned}\right\} \qquad (9.2)$$

The quantities to be considered have the general form

$$Q(\mathbf{x},t) = \psi^{\times}(\mathbf{x},t)\,O(\mathbf{x},t)\,\psi(\mathbf{x},t)\ . \qquad (9.3)$$

We shall investigate in the next section the case where the two spinors involved in Q are not the same. In (9.3), however, $\psi^{\times}$ is just $\bar{\psi}J^{-1}$, where $\bar{\psi}$ is the hermitian conjugate of ψ .

$$Q = \bar{\psi}\,J^{-1}\,O\,\psi\,. \qquad (9.4)$$

The requirement that Q be hermitian is equivalent to the requirement that $J^{-1}O$ be hermitian:

$$(J^{-1}O)^{-} = J^{-1}O \ ,$$

or
$$\rho_J = (J^{-1}OJ)/\bar{O} = +1 \ . \qquad (9.5)$$

We shall assume in the following that the operators O are so written that (9.5) is the case. The operators considered in the preceding sections are $J^{-1}O$ rather than O. See (2.40) in this connection.

To determine ρ_C, ρ_R and ρ_M, we use the auxiliary sign functions ρ_K, ρ_T and ρ_P defined by

$$\left.\begin{aligned} K^{-1}O(x,t)K &= \rho_K O^T(x,t) \ , & T^{-1}O(x,t)T &= \rho_T O(x,-t) \\ P^{-1}O(x,t)P &= \rho_P O(-x,t) \ , & J^{-1}O(x,t)J &= \rho_J \bar{O}(x,t) \end{aligned}\right\} \qquad (9.6)$$

See (7.17), (7.20). Then it is easy to see, by applying (9.1) on (9.2), that

$$\rho_C = \rho_K \quad , \quad \rho_R = -\rho_K \rho_T \quad , \quad \rho_M = -\rho_P \ . \qquad (9.7)$$

Note that $\rho_C = \rho_K$ is based on the anticommutability of ψ and ψ^x.

As an illustration, let us consider the free Hamiltonian H_0:

$$H_0 = E_0 E_\alpha p_\alpha + E_0 E_5 m \quad , \quad (\alpha = 1, 2, 3) \ .$$

The corresponding operator $O = JH_0$ is then

$$O = iE_\alpha p_\alpha + iE_5 m \ .$$

Since $J^{-1} E_i J = -\bar{E}_i$, (2.18), and $\bar{p}_\alpha = p_\alpha$, we have $\rho_J = +1$ on account of the imaginary unit i. Compare herewith that we had $\rho_J = -1$ for H_0, see Table I. Since O anticommutes with E_0, and since O does not contain the time variable, we have $\rho_T = -1$. Next, since $K^{-1} E_\mu K = -E_\mu^T$, $K^{-1} E_5 K = E_5^T$, (2.19), and $p_\alpha^T = -p_\alpha$, (2.71), we have $\rho_K = 1$ for O. Since $\Omega_x p_x = -p_x \Omega_x$, (2.70) and (2.74), and since E_{123} commutes with E_α and anticommutes with E_5, we have

$\rho_P = -1$. Therefore, by (9.7), $\rho_C = \rho_R = \rho_M = 1$, as should be.

We are mainly interested in the O's which are the basic E's or their products. There are, as well-known, six types of quantites up to the second rank:

$$\left.\begin{array}{llll} (S) & O = iE_5 , & (PS) & O = 1 , \\ (V) & O = iE^{\mu} , & (PV) & O = iE^{5\mu} , \\ (T) & O = E^{5\mu\nu} , & (PT) & O = iE^{\mu\nu} . \end{array}\right\} \tag{9.8}$$

The imaginary unit i is added wherever it is necessary to have $\rho_J = +1$. As we can write $E^{512} = iF_{30}$, etc., due to (2.17), the relation between (T) and (PT) is just that of complementary tensors as defined in Part I. Therefore, we need not investigate them separately. The various ρ's for the O's are listed in Table III, in which the indices α and β refer to space-coordinates: $\alpha, \beta = 1,2,3$.

ρ \ O	1	iE_5	iE_α	iE_0	$iE_{5\alpha}$	iE_{50}	$iE_{\alpha\beta}$	$iE_{\alpha 0}$
ρ_J	+	+	+	+	+	+	+	+
ρ_K	+	+	−	−	+	+	−	−
ρ_T	+	−	−	+	+	−	+	−
ρ_P	+	−	+	−	−	+	+	−
ρ_C	+	+	−	−	+	+	−	−
ρ_R	−	+	−	+	−	+	+	−
ρ_M	−	+	−	+	+	−	−	+
kind	1	reg.	2		1		3	

Table III. The various ρ's for the tensorial quantites built with the same spinors.

From the line for ρ_C, we observe that iE_μ, $iE_{\mu\nu}$ and $E_{5\mu\nu}$ are "electromagnetic" quantites ($\rho_C = -1$) while all the others are "mechanical" ($\rho_C = 1$).

From the lines for ρ_R and ρ_M, we can determine the "kinds" of the quantities, which are listed in the last line of the table. Comparing this result with (2.43), we observe that 1, iE_5 and $iE_{5\mu}$ change their "kinds" by the passage from the c-number theory to the q-number theory. This change is such that ρ_M remains the same while ρ_R changes its sign. It should also be noted that these quantities are "mechanical" quantities. In other words, the q-number theory changes the behavior of the mechanical quantities for reversion. These results were mentioned in Table VI of Part I.

It should also be recalled that the various alternative expressions for C, R and M pertain to the arbitrariness that is allowed without altering the transformation of the bilinear expression of the type (9.3). Therefore, the above mentioned results are not affected by the choices of the operators C, R and M.

The transformation rules (9.1) are determined by the consideration regarding $\mathcal{L}_\psi$ and s^μ. But, for the neutral spinor particles, s^μ does not have the meaning of electric current, and one may argue that particles and antiparticles are physically undistinguishable. If we take this point of view, we could define C, R and M by a combination of those C, R and M given in (9.1) with charge-conjugation $\mathcal{C}$: $\psi \rightarrow \psi^\times K$, $\psi^\times \rightarrow \psi K^{-1}$, i.e., by

$$\left.\begin{aligned} C\,\psi\, C^{-1} &= e^{i\alpha}\psi \\ (R^{-1}\psi(t)R)^T &= e^{i\beta}E_0\psi(-t) \\ M\,\psi(x)\,M^{-1} &= e^{i\gamma}E_{123}\psi^\times(-x)K \end{aligned}\right\} \qquad (9.9)$$

But such an assumption will immediately encounter a serious difficulty. Consider the pion-nucleon interaction:

$$N + \pi^+ \rightarrow P\,, \qquad (9.10)$$

to which corresponds an interaction Hamiltonian of the type

$$\psi_P^\times \psi_N \upsilon \, , \qquad (9.11)$$

where the Dirac matrices are omitted for simplicity. Assuming (9.1) for protons and (9.9) for neutrons, we get as the result of mirage, for instance,

$$\psi_P^\times \psi_N^\times \upsilon \, , \qquad (9.12)$$

which certainly should not exist in the original interaction Hamiltonian, since it implies a violation of "conservation of heavy particles":

$$\pi^+ \to P + N \, . \qquad (9.13)$$

The relations (9.9) for C and R equally encounter the same difficulties. Therefore we have to assume (9.1) and not (9.9) for neutrons.

We can further argue that the neutrino has also to obey (9.1) and not (9.9). For if the neutrino would obey (9.9), the interaction Hamiltonian of the type

$$P \to N + e^+ + \bar{\nu} \; : \quad (\psi_N^\times \psi_P)(\psi_e^\times \psi_\nu) \, , \qquad (9.14)$$

would go over, by mirage, to

$$P \to N + e^+ + \nu \; : \quad (\psi_N^\times \psi_P)(\psi_e^\times \psi_\nu^\times) \, , \qquad (9.15)$$

where the neutron, proton and electron are assumed to obey (9.1). The combination of (9.14) and (9.15) would mean

$$\nu \to \bar{\nu} \quad \text{or} \quad \nu + \nu \to 0 \, , \qquad (9.16)$$

which is absurd.

For these reasons, we shall assume in the following sections that the neutron as well as neutrino obeys (9.1).

#10. Nucleon-Pion Interaction

Let us consider the nucleon-pion interaction of the type

$$\mathcal{L} = f\, \psi^{\times} O_{\alpha} \vec{\tau}\, \psi\, \vec{F}^{\alpha} , \tag{10.1}$$

where the index α designates the tensorial components and the arrow means a vector in the 3-dimensional Euclidean isotopic spin space. ψ is actually an 8-component spinor composed of neutron spinor ψ_N and proton spinor ψ_P , which we define by

$$\tau_3 \psi_N = + \psi_N \quad , \quad \tau_3 \psi_P = - \psi_P \; . \tag{10.2}$$

The factor $\vec{F}^{\alpha}$ is a hermitian quantity derived from the pion field and the τ's are the well-known isotopic spin matrices which are also hermitian. Then, if we take (which we do) O such that

$$J^{-1} O J = \bar{O} \quad , \quad \text{i.e.} \quad \rho_J = 1 , \tag{10.3}$$

$\mathcal{L}$ will become hermitian with real f.

Although we take the "symmetric" formulation, (10.1), as the standard expression, it will be easy to infer the situation with the "charged" or "neutral" formulation from our results. As O in (10.1), we can take any one of the operators introduced in (9.8). The $\vec{F}^{\alpha}$ for (S) and (PS) is the scalar pion field strength itself $\vec{u}$. The $\vec{F}^{\alpha}$ for (V) and (PV) can be the vectorial field strength $\vec{u}^{\mu}$ or the 4-dimensional gradient of the scalar field $\vec{u}$. The $\vec{F}^{\alpha}$ for (T) and (PT) must be the 4-dimensional curl of a vectorial field $\vec{u}^{\mu}$.

In this section, we take the view point that ψ_N and ψ_P obey the transformation rules (9.1) and that the transformation rules of the u-field are to be determined by the requirement that $\mathcal{L}$ (10.1) be a regular scalar. There is no reason why the phase angles α , β and γ should be the same for protons and neutrons. Therefore, we shall write α_P , β_P , γ_P for protons and

α_N, β_N and γ_N for neutrons.

By decomposing (10.1) in three components in the isotopic spin space, we get

$$\left.\begin{aligned} \mathcal{L} &= f\,(\psi_N^\times O_\alpha \psi_P + \psi_P^\times O_\alpha \psi_N)\,F^{(1)\alpha} \\ &+ f\,(-i\psi_N^\times O_\alpha \psi_P + i\,\psi_P^\times O_\alpha \psi_N)\,F^{(2)\alpha} \\ &+ f\,(\psi_N^\times O_\alpha \psi_N - \psi_P^\times O_\alpha \psi_P)\,F^{(3)\alpha} \end{aligned}\right\} \qquad (10.4)$$

Or by introducing, in analogy to (5.1),

$$G^\alpha = (F^{(1)\alpha} + i\,F^{(2)\alpha})/\sqrt{2}\,, \qquad \bar{G}^\alpha = (F^{(1)\alpha} - i\,F^{(2)\alpha})/\sqrt{2}\,,$$

we can write

$$\mathcal{L} = \sqrt{2}f\,(K_\alpha G^\alpha + \bar{K}_\alpha \bar{G}^\alpha) + f\,H_\alpha F^{(3)\alpha} \qquad (10.5)$$

with

$$\left.\begin{aligned} K_\alpha &= \psi_P^\times O_\alpha \psi_N\,, \\ \bar{K}_\alpha &= \psi_N^\times O_\alpha \psi_P\,, \\ H_\alpha &= \psi_N^\times O_\alpha \psi_N - \psi_P^\times O_\alpha \psi_P\,. \end{aligned}\right\} \qquad (10.6)$$

In the "charged" formulation, the third term with $F^{(3)}$ is absent. In the "neutral" formulation, the first two terms are absent and we have a plus sign instead of minus sign in the expression of H_α.

Our next step is to determine the transformation rules of K_α and H_α by the use of (9.1). The results are as follows:

$$\left.\begin{aligned} C K_\alpha C^{-1} &= [\pm]\, e^{i(\alpha_N - \alpha_P)}\, \rho_C \bar{K}_\alpha \\ C H_\alpha C^{-1} &= \rho_C H_\alpha \end{aligned}\right\} \qquad (10.7)$$

$$\left.\begin{aligned} (R^{-1} K_\alpha(t) R)^T &= e^{i(\beta_N - \beta_P)} \cdot \rho_R \bar{K}_\alpha(-t) \\ (R^{-1} H_\alpha(t) R)^T &= \rho_R H_\alpha(-t) \end{aligned}\right\} \qquad (10.8)$$

$$\left.\begin{aligned} M K_\alpha(x) M^{-1} &= e^{i(\gamma_N - \gamma_P)} \cdot \rho_M K_\alpha(-x) \\ M H_\alpha(x) M^{-1} &= \rho_M H_\alpha(-x) \end{aligned}\right\} \qquad (10.9)$$

The sign $[\pm]$ is positive if ψ_N and ψ_P anticommute and negative if they commute. ρ_C , ρ_R and ρ_M are the ones determined in the last section for various 0's, see Table III.

By the requirement that $\mathcal{L}$ be a regular scalar, we can infer the transformation rules for $F^{(3)}$ from the results (10.7), (10.8) and (10.9):

$$C F^{(3)\alpha} C^{-1} = \rho_C F^{(3)\alpha} \quad , \quad (R^{-1} F^{(3)\alpha}(t)\, R)^T = \rho_R\, F^{(3)\alpha}(-t) \; , \quad M F^{(3)\alpha}(x) M^{-1} = \rho_M F^{(3)\alpha}(-x) , \tag{10.10}$$

where ρ_C , ρ_R and ρ_M are those of the operator 0 used in (10.1). Since the $F^{(3)}$ is the $u^{(3)}$ itself, or a quantity derived from $u^{(3)}$ by an operation of differentiation which is a regular tensor, the "kind" of $F^{(3)}$ is the same as the kind of $u^{(3)}$. Also, the charge-sign-function ρ_C must be the same for $F^{(3)}$ and $u^{(3)}$. These properties of $u^{(3)}$ are tabulated in Table IV.

	(S)	(V)	(T)	(PS)	(PV)	(PT)
kind	reg.	2	2	1	1	3
ρ_C	+	−	−	+	+	−

Table IV. The transformation properties of $u^{(3)}$ for various interaction types.

From these properties, we can conclude an important rule concerning the mixing of different types of interaction: Mixing of (S) and (V) for scalar field and mixing of (PV) and (PT) for vector field are not allowed. This conclusion can be drawn from both consideration of the "kind" as well as from consideration of ρ_C . This conclusion is also valid in the "neutral formulation".

Contrary to the case of $u^{(3)}$, we cannot infer any definite conclusion regarding the "kind" of v and $\bar{v}$ because of the unknown factor such as $\exp(i\,\gamma_N - i\,\gamma_P)$ in (10.9). By the same argument as the one which led to

(10.10) we can write

$$\left.\begin{aligned} C \upsilon C^{-1} &= e^{i(\alpha_P-\alpha_N)} \rho_C \bar{\upsilon} \\ (R^{-1} \upsilon(t) R)^T &= e^{i(\beta_P-\beta_N)} \rho_R \bar{\upsilon}(-t) \\ M \upsilon(x) M^{-1} &= e^{i(\gamma_P-\gamma_N)} \rho_M \upsilon(-x) \end{aligned}\right\} \quad (10.11)$$

where ρ_C, ρ_R and ρ_M are those of $u^{(3)}$ for the corresponding interaction type. These relations agree with (5.8), (5.11) and (5.16). Since γ_P and γ_N are limited to $\pm\pi/4$, $\exp(i\gamma_P - i\gamma_N)$ as well as $\rho_M \exp(i\gamma_P - i\gamma_N)$ is limited to ± 1, which is in agreement with (5.17).

The prohibition rules of mixture, explained with regard to $u^{(3)}$, is still valid for the charged υ. For, whatever the factors $\exp(i\alpha_P - i\alpha_N)$ and $\exp(i\beta_P - i\beta_N)$ may be, the phase factors in transformation rules for υ must take the same values throughout the entire Lagrangean. It is true that on account of the so-called equivalence theorem,[28] the (S)-interaction and (V)-interaction, for instance, are equivalent. Thus one may say that the mixture of (S) and (V) is anyway unnecessary. However, these equivalence theorems are proved only in lower approximations and we cannot in general replace (S) by (V) or (V) by (S).

Before closing this section, a few lines may be spent in connection with the contention often raised[29] to the effect that the (S,V,T)-types can be mixed with (PS, PV, PT)-types. For simplicity, let us consider the mixture of (S) and (PS), in accordance with the usually proposed procedure:

$$\mathcal{L} = f_1 \psi^{\times} \vec{\tau} \psi \vec{u} + (1/24) f_2 \psi^{\times} E_{\mu\nu\kappa\lambda} \vec{\tau} \psi \eta^{\mu\nu\kappa\lambda} \vec{u}, \qquad (10.12)$$

where $E_{\mu\nu\kappa\lambda} = \pm iE_5$ according as (μ,ν,κ,λ) is an even or odd permutation of

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[28] E.C. Nelson, Phys. Rev. 60, 830(1941); E.J.Dyson, Phys.Rev. 73, 929(1948); K.M.Case, Phys.Rev., 76, 14(1949).

[29] M. Schoenberg, Phys.Rev. 60, 468(1941).

(1,2,3,0). See (2.41). According to the c-number theory, $\psi^{\times} E_{\mu\nu\kappa\lambda}\psi$ is a 3rd kind pseudo-tensor of the fourth rank, corresponding to the fact that $\psi^{\times} E_5 \psi$ which is the complementary tensor of $\psi^{\times} E_{\mu\nu\kappa\lambda}\psi$ is a 2nd kind pseudo-scalar. In the q-number theory, $\psi^{\times} E_{\mu\nu\kappa\lambda}\psi$ is 1st kind and $\psi^{\times} E_5 \psi$ is regular.

Now referring to (10.12), the first term will be a regular sclar if $u^{(3)}$ is 3rd kind in c-number theory and 1st kind in q-number theory and v obeys the corresponding relation in (10.11). Then the second term in (10.12) can be made regular, if we assume $\eta^{\mu\nu\kappa\lambda}$ to be regular in both c-number and q-number theories. For, (3rd kind) x (regular) x (3rd kind) = regular, and (1st kind) x (regular) x (1st kind) = regular.

This regular tensor $\eta^{\mu\nu\kappa\lambda}$ is nothing but what we denoted by the same symbol in Section 2, Part I, as distinct from $\epsilon^{\mu\nu\kappa\lambda}$. However, the appearance of $\eta^{\mu\nu\kappa\lambda}$ in a physical law is highly questionable. For, η^{1230}, for instance, is $+1$ in one coordinate system, but is -1 in another coordinate system which is connected to the first by mirage. This means that the physical law is different for the right-handed coordinate system and the left-handed system. How can one decide whether he should take $\eta^{1230} = +1$ or $= -1$ in his coordinate system? In other words, we can write the second term of (10.12) in the form

$$f_2\, i\, \psi^{\times} E_{\mu} \vec{\tau}\, \psi\, \vec{u}\,, \tag{10.13}$$

assuming that f_2 is a first kind pseudo-scalar. Is it permissible that a natural constant changes its sign by mirage or reversion? Can it differ for a process and its miraged or reversed process?

Furthermore, mixture of one of (S), (PS), (PV) with one of (V)(T)(PT) cannot be made allowable by a similar artifice, because ρ_C is different for these two groups.

#11. Nucleon-Lepton Interaction

The usual nucleon-lepton interaction Lagrangean can be written

$$\mathcal{L} = \sum_\alpha f_\alpha (\psi_N^\times O_\alpha \psi_P)(\psi_e^\times O^\alpha \psi_\nu) + \sum_\alpha f_\alpha^* (\psi_P^\times O_\alpha \psi_N)(\psi_\nu^\times O^\alpha \psi_e) \ , \tag{11.1}$$

where the "particle", as distinguished from "antiparticle", of the ψ_e- field is positron, and the "particle" of the ψ_ν- field is the neutrino as defined as the neutral partner of the negative beta-decay. The index α designates five independent types of operators mentioned in (9.8). Because of the existing condition (9.5), $\mathcal{L}$ is hermitean if f_α^* is the complex conjugate of f .

By applying the first transformation of (9.1) on the first term of (11.1) we obtain

$$\begin{aligned} & f_\alpha C (\psi_N^\times O_\alpha \psi_P)(\psi_e^\times O^\alpha \psi_\nu) C^{-1} \\ & = f_\alpha (\psi_N O_\alpha^T \psi_P^\times)(\psi_e O^{\alpha T} \psi_\nu^*) \, e^{i(\alpha_P - \alpha_N + \alpha_\nu - \alpha_e)} \quad (\rho_c)^2 \end{aligned} \tag{11.2}$$

where $\rho_c (= \pm 1)$ refers to O_α , therefore $\rho_c^2 = 1$. It may be quite natural to assume that the commutation relations between ψ_N and ψ_P and the commutation relation between ψ_ν and ψ_e are of the same type. Therefore the q-number part of (11.2) is identical with the q-number part of the second term of (11.1). Therefore we get the relation

$$f_\alpha^* = f_\alpha \, e^{i(\alpha_P - \alpha_N + \alpha_\nu - \alpha_e)} \ , \tag{11.3}$$

or

$$f_\alpha = |f_\alpha| \, e^{i\theta_\alpha} \ , \tag{11.4}$$

with

$$\theta_\alpha = (-\alpha_P + \alpha_N - \alpha_\nu + \alpha_e + 2 n_\alpha \pi)/2 \ , \quad (n_\alpha = 0, 1, 2 \cdots) . \tag{11.5}$$

By the same token, from reversion and mirage, we obtain (without any assumption regarding the type of commutation rules)

$$f_\alpha^* = f_\alpha \, e^{i(\beta_P - \beta_N + \beta_\nu - \beta_e)} \tag{11.6}$$

and $$f_\alpha = f_\alpha \, e^{i(\gamma_P - \gamma_N + \gamma_\nu - \gamma_e)} \quad . \tag{11.7}$$

The compatibility of (11.3) with (11.6) requires

$$\alpha_P - \alpha_N + \alpha_\nu - \alpha_e = \beta_P - \beta_N + \beta_\nu - \beta_e + 2n\pi , \; (n = 0, 1, 2, \dots) , \tag{11.8}$$

and the self-consistency of (11.7) requires

$$\gamma_P - \gamma_N + \gamma_\nu - \gamma_e = 2n\pi \qquad (n = 0, 1, 2, \dots) , \tag{11.9}$$

while individual γ's are $\pm \pi/4$.

Whether or not $\exp(i\theta_\alpha)$ separately for various α's have a physical meaning, the relative phases $\exp i(\theta_\alpha - \theta_{\alpha'})$ have certainly a physically detectable effect.[30] From (11.5) follows

$$\theta_\alpha - \theta_{\alpha'} = (n_\alpha - n_{\alpha'})\pi \tag{11.10}$$

or $$e^{i(\theta_\alpha - \theta_{\alpha'})} = \pm 1 \tag{11.11}$$

This means that for a mixture of the type (11.1), the f's can be complex but the ratio $f_\alpha / f_{\alpha'}$ for all combinations of α and α' must be real.[30] Obviously the same conclusion can be drawn from the consideration of reversion, i.e., from (11.6).

Since the transformation of a quantity of the type $(\psi_1^\times O \psi_2)$ contains an undetermined factor, such as $\exp[i\beta_2 - i\beta_1]$, we cannot maintain any longer the old prescription that a tensor (pseudo-tensor) of nucleon field should not be coupled with a pseudo-tensor (tensor) of lepton field. Therefore we can assume,[31] instead of (11.1),

$$\mathcal{L} = \sum_\alpha h_\alpha (\psi_N^\times O_\alpha \psi_P)(\psi_e^\times O^{5\alpha} \psi_\nu) + \sum_\alpha h_\alpha^* (\psi_P^\times O_\alpha \psi_N)(\psi_\nu^\times O^{5\alpha} \psi_e) , \tag{11.12}$$

where $O^{5\alpha}$ means E_5 times or iE_5 times O. Table V gives the exact correspondence.

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[30] L.C.Biedeharn and M.E.Rose, Phys.Rve 83, 459(1951)
[31] C.N.Yang and J.Tiomno, Phys.Rev 79, 495(1950)

O_α	iE_5	1	iE_μ	$iE_{5\mu}$	$E_{5\mu\nu}$	$iE_{\mu\nu}$
$O^{5\alpha}$	1	iE^5	$iE^{5\mu}$	iE^μ	$iE^{\mu\nu}$	$E^{5\mu\nu}$

Table V. Correspondence of $O^{5\alpha}$ to E_α .

By the same argument as before, we obtain from charge-conjugation, reversion and mirage, the following results:

$$\left.\begin{aligned} h_\alpha^* &= e^{i(\alpha_p-\alpha_N+\alpha_\nu-\alpha_e)}\,\rho_C(O_\alpha)\rho_C(O_{5\alpha})\,h_\alpha \\ h_\alpha^* &= e^{i(\beta_p-\beta_N+\beta_\nu-\beta_e)}\,\rho_R(O_\alpha)\rho_R(O_{5\alpha})\,h_\alpha \\ h_\alpha &= e^{i(\gamma_p-\gamma_N+\gamma_\nu-\gamma_e)}\,\rho_M(O_\alpha)\rho_M(O_{5\alpha})\,h_\alpha \end{aligned}\right\} \tag{11.13}$$

where ρ_C , ρ_R and ρ_M are the ones given in Table III. The products such as $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ are listed in Table VI, in which the first line indicates the operator O_α .

O	(S)	(PS)	(V)	(PV)	(T)	(PT)
$\rho_C(O_\alpha)\rho_C(O_{5\alpha})$	+	+	−	−	+	+
$\rho_R(O_\alpha)\rho_R(O_{5\alpha})$	−	−	+	+	−	−
$\rho_M(O_\alpha)\rho_M(O_{5\alpha})$	−	−	−	−	−	−
class	I	I	II	II	I	I

Table VI. The values of $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$, etc., as dependent on the type of O_α .

In the first place, we notice that $\rho_M(O_\alpha)\rho_M(O_{5\alpha})$ is always −1. Therefore from the third relation of (11.13), we have

$$\gamma_p-\gamma_N+\gamma_\nu-\gamma_e=(2n+1)\pi \tag{11.14}$$

Regarding $\rho_C(O_\alpha)\rho_C(O_{5\alpha})$ and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$, we observe that the types of combinations can be divided into two classes:

$$\left.\begin{array}{ll}\text{(I)} & (S)-(PS)\ ,\ (PS)-(S)\ ,\ (T)-(PT)\ ,\ (PT)-(T)\ ; \\ \text{(II)} & (V)-(PV)\ ,\ (PV)-(V)\ . \end{array}\right\} \tag{11.15}$$

For class (I), $\rho_c(O_\alpha)\rho_c(O_{5\alpha})$ is $+1$, and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$ is -1. For class (II), $\rho_c(O_\alpha)\rho_c(O_{5\alpha})$ is -1 and $\rho_R(O_\alpha)\rho_R(O_{5\alpha})$ is $+1$.

Writing

$$A = \alpha_P - \alpha_N + \alpha_\nu - \alpha_e \ , \quad B = \beta_P - \beta_N + \beta_\nu - \beta_e \tag{11.16}$$

we have for class (I),

$$h_\alpha = |h_\alpha| e^{i\theta_\alpha} \tag{11.17}$$

with
$$\theta_\alpha = -\frac{1}{2} A + n_\alpha \pi = -\frac{1}{2} B + (m_\alpha + \frac{1}{2})\pi \tag{11.18}$$

And for class II

$$\theta_\alpha = -\frac{1}{2} A + (n_\alpha + \frac{1}{2})\pi = -\frac{1}{2} B + m_\alpha \pi \tag{11.19}$$

From (11.18) follows that if α and α' belong both to I or both to II

$$e^{i\theta_\alpha}/e^{i\theta_{\alpha'}} = \pm 1 \qquad (\alpha, \alpha' \in \mathrm{I} \text{ or } \alpha, \alpha' \in \mathrm{II}) \tag{11.20}$$

and that if one of α and α' belongs to I and the other belongs II,

$$e^{i\theta_\alpha}/e^{i\theta_{\alpha'}} = \pm i \qquad (\alpha \in \mathrm{I}, \alpha' \in \mathrm{II} \text{ or } \alpha \in \mathrm{II}, \alpha' \in \mathrm{I}) \tag{11.21}$$

In other words, within the same class, I or II, the relative phase-factors of h's are real, and the relative phase-factors of the h's belonging to the different classes, I and II, are imaginary.

It should also be noted that a mixture of a type belonging to (11.1) and a type belonging to (11.12) is not allowed. For (11.9) and (11.14) are in a direct comtradiction. The same conclusion can also be drawn from the comparison of (11.8) with (11.18) or (11.19).

#12. Selection Rules and Superselection Rules

A self-miraged state Ψ is a state such that the expectation values of each physical quantity $Q(x)$ is ρ_M times that of $Q(-x)$. In particular, a quantity Q, which does not depend on x and whose ρ_M is negative, must have a vanishing expectation value. The total linear momentum is an example of such a quantity.

By definition of the mirage operator M, we should have for a self-miraged state

$$M\Psi = e^{i\alpha}\Psi \; ; \quad M^2\Psi = e^{2i\alpha}\Psi .$$

If

$$M^2 = 1 , \tag{12.1}$$

we have $\exp(2i\alpha) = 1$ and the self-miraged states are classified into states of even parity and odd parity:

$$\left.\begin{aligned} M\Psi_{even} &= \Psi_{even} \\ M\Psi_{odd} &= -\Psi_{odd} \end{aligned}\right\} \tag{12.2}$$

However, this definition is not yet unique, because we can still multiply M by (-1) without changing (12.1) but interchanging Ψ_{even} and Ψ_{odd}. Therefore, we usually impose an auxiliary definition.

$$M\Psi_{vac} = +\Psi_{vac} \tag{12.3}$$

which is just a matter of convention.

In order that this classification (12.2) is compatible with the other specification of Ψ, we have to use those physical quantities which commute with M or whose expectation values in Ψ are zero. For those quantities Q which do not depend on X, the relation (3.12) simply means that $\rho_M = 1$ or -1 according as Q commutes or anticommutes with M. Therefore, Ψ must have vanishing expectation values for these Q's with $\rho_M = -1$. This is just what a

self-miraged state is.

This restriction to "self-miraged" states would seem to delimit the Hilbert space to its part or subspace. But it is not so because any arbitrary state can be decomposed into two kinds of self-miraged states (12.2).

$$\Psi = \Psi_{even} + \Psi_{odd} \quad ; \quad \Psi_{even} = (\Psi + M\Psi)/2 \quad ; \quad \Psi_{odd} = (\Psi - M\Psi)/2 \ , \tag{12.4}$$

where the resultant Ψ is not self-miraged.

In linear momentum representation, the self-miraged states must have the total linear momentum equal to zero. We have already noticed that the angular momentum representation is particularly suitable for the parity consideration. It should also be recalled that except for the case of photons and neutral mesons, there is actually no physical means of determining the parity of a single particle. This ambiguity is one which exists even after the restrictions (12.1) and (12.3) are made. The situation has been explained in some detail with regard to the charged spinor fields in the angular momentum representation. A similar consideration applies also to the charged pion fields.

The selection rule regarding space-parity means that the transition between an even-parity state and an odd-parity state is prohibited if the transition operator $U(t \quad -t)$ commutes with the mirage operator M. This is quite obvious, because

$$\begin{aligned}(\Psi_{odd}, U\Psi_{even}) &= (\Psi_{odd}, UM\Psi_{even}) \\ &= (\bar{M}\Psi_{odd}, U\Psi_{even}) = -(\Psi_{odd}, U\Psi_{even})\end{aligned} \tag{12.5}$$

We can do exactly the same thing with the charge conjugation operator C:

$$\left.\begin{aligned} &C^2 = 1 \ ; \qquad C\Psi_{vac} = \Psi_{vac} \ ; \\ &C\Psi^{ch}_{even} = \Psi^{ch}_{even} \quad ; \qquad C\Psi^{ch}_{odd} = -\Psi^{ch}_{odd} \ ; \\ &(\Psi^{ch}_{odd}, U\Psi^{ch}_{even}) = 0 \ \text{ since } \ UC - CU = 0 \ . \end{aligned}\right\} \tag{12.6}$$

Since we have the same (x, t) on the left and right sides of (3.14), the self-change-conjugate states must have vanishing expectation values of all "electromagnetic" quantities ($\rho_c = -1$). By a superposition of a state with even charge-parity and a state with odd charge-parity, we can build a state in which "electromagnetic" quantities have non-vanishing expectation values.

To have a concise view of the entire problem and also to prepare a step toward the consideration of superselection rules, let us consider a unitary or hermitian operator W which has eigen-values W_i and corresponding eigen-functions Ψ_i:

$$W \Psi_i = W_i \Psi_i , \tag{12.7}$$

where we use the same value of i insofar as W_i is the same. In other words, Ψ_i determines in general a sub-space in the Hilbert space. Now, take any operator X that commutes with W:

$$XW = WX , \tag{12.8}$$

then we have

$$(\Psi_j, XW\Psi_i) = (\Psi_j, WX\Psi_i) = (\overline{W}\Psi_j, X\Psi_i),$$

from which follows

$$\left.\begin{aligned} &W_i(\Psi_j, X\Psi_i) = (1/W_j^*)(\Psi_j, X\Psi_i) \qquad &\text{(W: unitary)} \\ \text{or} \quad &W_i(\Psi_j, X\Psi_i) = W_j^*(\Psi_j, X\Psi_i) \qquad &\text{(W: hermitian)} \end{aligned}\right\} \tag{12.9}$$

Putting first $X = 1$, we get the well-known theorems that the eigen-values of a unitary operator are of absolute value unity and those of a hermitian operators are real, and that in either case Ψ_i and Ψ_j are orthogonal if $W_i \neq W_j$:

$$(\Psi_i, \Psi_j) = 0 \quad , \quad i \neq j . \tag{12.10}$$

Further putting $X = U$, we get

$$(\Psi_i, U\Psi_j) = 0 \quad , \quad i \neq j \tag{12.11}$$

on condition that

$$UW = WU. \tag{12.12}$$

This leads to the "conservation law" of the eigen value. The selection rules with regard to space-parity and charge-parity can be obtained by putting respectively $W = M$ and $W = C$.

The concept of a superselection rule is more restringent than that of a selection rule. Namely, if there is a unitary or hermitian operator W that commutes not only with U but also all physical quantities Q, then we speak of a superselection rule. Under this condition, we shall have

$$QW = WQ \tag{12.13}$$

and, putting $X = Q$ we get from (12.9),

$$(\Psi_i, Q\Psi_j) = 0 \quad , \quad i \neq j. \tag{12.14}$$

If W commutes with all Q, then it will commute with the Hamiltonian and also with the transition matrix. Therefore, if (12.13) is the case, it is not necessary to require (12.12) separately. (12.14) means that the expectation value of Q in a state:

$$\Psi = \sum a_i \Psi_i \tag{12.15}$$

is given by

$$(\Psi, Q\Psi) = \sum_i |a_i^2| (\Psi_i, Q\Psi_i) \tag{12.16}$$

This shows that the phase of a_i loses any physical meaning. In other words,

the knowledge of a state expressed by (12.15) is nothing better than a mixed density G

$$G = \sum |a_i^2| \mathcal{E}[\Psi_i] \tag{12.17}$$

where $\mathcal{E}[\Psi_i]$ is the projection operator onto the direction of Ψ_i :

$$\mathcal{E}[\Psi_i]\Psi_i = \Psi_i \quad , \quad \mathcal{E}[\Psi_i]\Phi = 0 \quad , \tag{12.18}$$

where Φ is any state-function orthogonal to Ψ_i.

Furthermore, it should be noted that a state Ψ and $W\Psi$ are physically indistinguishable if W is unitary:

$$(W\Psi, QW\Psi) = (\Psi, \overline{W}QW\Psi) = (\Psi, Q\Psi). \tag{12.19}$$

The simplest way to discover superselection rules is to search for gauge transformations under which all the known physical quantities remain invariant. We have seen in the body of this paper that the sign of electromagnetic field and $\mathcal{U}^{(3)}$-field seem to have physical meaning. All the rest of the field strengths are open to unobservable phase-change. The physical quantities which refer separately to a spinor field ψ and a charged boson field $\mathcal{V}$ have the form

$$\psi^\times \psi \qquad \text{and} \qquad \overline{\mathcal{V}}\mathcal{V} , \tag{12.20}$$

insofar as their dependence on ψ and $\mathcal{V}$ is concerned. The electromagnetic interaction has a term like $\psi_e^\times \psi_e A$ or $\psi_P^\times \psi_P$ A, where A stands for electromagnetic potential and the suffices e and P refer to (positive) electron and proton. But, as A is not changed by a gauge transformation with a constant phase, these quantities can be included in (11.20). The interaction Hamiltonians between nucleon fields, lepton fields and boson fields are of the type

$$\psi_N^\times \psi_P \overline{\mathcal{V}} \quad , \quad \psi_N^\times \psi_P \psi_e^\times \psi_\nu . \tag{12.21}$$

See Sections 10, 11.

One of the simplest gauge transformations that leave (11.20) and (11.21) is:

$$\psi \to e^{i\alpha}\psi \quad , \quad \psi^{\times} \to e^{-i\alpha}\psi^{\times} \tag{12.22}$$

for all the spinors. It is very tempting to try the same thing with respect to υ, i.e., to assume $\upsilon \to e^{i\beta}\upsilon$. However, in this case, we have to take into consideration a term like $\psi_N^{\times}\psi_P\bar{\upsilon}$. If $\bar{\upsilon} \to e^{-i\beta}\bar{\upsilon}$, then $\psi_N^{\times}\psi_P$ must go over to $e^{i\beta}\psi_N^{\times}\psi_P$. And furthermore, if $\psi_N^{\times}\psi_P \to e^{i\beta}\psi_N^{\times}\psi_P$, then from a quantity like $\psi_N^{\times}\psi_P\psi_e^{\times}\psi_\nu$, we conclude that $\psi_e^{\times}\psi_\nu$ should pass to $e^{-i\beta}\psi_e^{\times}\psi_\nu$. All this chain of transformations can be written as follows

$$\left.\begin{array}{llll} \upsilon \to e^{i\beta} & , & \bar{\upsilon} \to e^{-i\beta}\bar{\upsilon} & , \\ \psi_N \to e^{i\gamma - i\beta/2}\psi_N & , & \psi_P \to e^{i\gamma + i\beta/2}\psi_P & , \\ \psi_\nu \to e^{i\delta - i\beta/2}\psi_\nu & , & \psi_e \to e^{i\delta + i\beta/2}\psi_e & , \end{array}\right\} \tag{12.23}$$

where β, γ, δ are entirely arbitrary except that they are real. If we put $\beta = 0$, $\gamma = \delta$, we come back to (11.23). If we put $\gamma = \delta = \beta/2$, then we get the familiar gauge transformation in which all the positive fields are multiplied by $e^{i\beta}$, the negative fields by $e^{-i\beta}$, and the neutral fields unchanged. If we put $\gamma = \delta = 0$, we get a transformation which can be interpreted as a rotation by angle β about the third axis of the isotopic spin. This last transformation is obtained by considering ψ in $\psi^{\times}\vec{\tau}\psi\,\vec{u}$ as a spinor in the three dimensional space in which $\vec{\tau}$ plays the role of spin matrices and by applying (2.8) to $\vec{u}$ and (2.57) to ψ.

In (12.23) the wave-fields are not yet separated into "positive" waves and "negative" waves, or into "particle" waves and "anti-particle" waves. To avoid confusion in doing this separation, let us make some agreement about symbols: We use (+) and (–), to designate positive and negative charge or to designate

"particle" and "antiparticle". Thus, (+) will include proton, positive electron, neutron, neutrino and positive pion, while (-) will include negative proton, negative electron, anti-neutron, anti-neutrino and negative pion. We shall use (o) to indicate neutral fermions. Furthermore, (h) and (ℓ) will differentiate the heavy particle (nucleon) family from the light particle (lepton) family. With this convention about the symbols, we can rewrite (12.23) as follows.

$$\left.\begin{aligned} &\upsilon_\pm \to e^{\pm i\beta}\,\upsilon_\pm\,, \\ &\psi^h_\pm \to e^{\pm i(\gamma+\beta/2)}\,\psi^h_\pm\,, \qquad \psi^h_{o\pm} \to e^{\pm i(\gamma-\beta/2)}\,\psi^h_{o\pm} \\ &\psi^\ell_\pm \to e^{\pm i(\delta+\beta/2)}\,\psi^\ell_\pm\,, \qquad \psi^\ell_{o\pm} \to e^{\pm i(\delta-\beta/2)}\,\psi^\ell_{o\pm} \end{aligned}\right\} \tag{12.24}$$

Now we re-interpret these relations in the quantized field theory by expanding the υ's and the ψ's in the absorption operators and regarding the transformations in (12.24) as a unitary transformation, e.g.,

$$W\,\upsilon_+\,W^{-1} = e^{i\beta}\,\upsilon_+\,, \qquad \text{etc.} \tag{12.25}$$

Such a W is very easily found[32] to be

$$\begin{aligned} W = \prod \left(e^{-i\beta}\right)^{M_+-M_-} \left(e^{-i(\gamma+\beta/2)}\right)^{N^h_+-N^h_-} \left(e^{-i(\gamma-\beta/2)}\right)^{N^h_{o+}-N^h_{o-}} \\ \left(e^{-i(\delta+\beta/2)}\right)^{N^\ell_+-N^\ell_-} \left(e^{-i(\delta-\beta/2)}\right)^{N^\ell_{o+}-N^\ell_{o-}} \end{aligned} \tag{12.26}$$

where the N's and the M's stand for occupation number operators respectively for fermion eigen-states and boson eigen-states. The multiplication $\prod$ should extend over all the possible eigen-states.

To make the implication of (12.26) more transparent, let us break it down to three independent cases: (1) $\gamma=\delta=0$, (2) $\beta=\delta=0$, and (3) $\beta=\gamma=0$. In the first case we have

$$W = \prod \left(e^{-i\beta/2}\right)^{2M_+-2M_-+N_+-N_--N_{o+}+N_{o-}}\,, \tag{12.27}$$

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[32] S. Watanabe, Phys. Rev. in press.

where the N's stand for both (h) and (ℓ). The eigen values of W(12.27) are $(e^{-i\beta/2})^r$, $r = 0, \pm 1, \pm 2, \ldots$, and the selection rule engendered by this W means the conservation of r, i.e., conservation of

$$\sum (2M_+ - 2M_- + N_+ - N_- - N_{0+} + N_{0-}) = \text{constant} . \tag{12.28}$$

In the second case (2), we have

$$W = \prod (e^{-i\gamma})^{N_+^h - N_-^h + N_{0+}^h - N_{0-}^h} , \tag{12.29}$$

from which follows the conservation law of

$$\sum (N_+^h - N_-^h + N_{0+}^h - N_{0-}^h) = \text{constant} . \tag{12.30}$$

In the case (3), we have

$$W = \prod (e^{-i\delta})^{N_+^\ell - N_-^\ell + N_{0+}^\ell - N_{0-}^\ell} , \tag{12.31}$$

and

$$\sum (N_+^\ell - N_-^\ell + N_{0+}^\ell - N_{0-}^\ell) = \text{constant} . \tag{12.32}$$

It is to be noted that (12.30) and (12.32) can exclude the undesired transitions in nucleon-pion interaction and nucleon-lepton interaction,[33] such as

$$N + P \rightarrow \pi^+ ,$$

$$N + P \rightarrow \nu + e^+ ,$$

$$N + P \rightarrow \bar{\nu} + e^+ .$$

By adding the three conservation laws: (12.28), (12.30) and (12.32), we get

$$M_+ - M_- + N_+ - N_- = \text{constant} , \tag{12.33}$$

which is nothing but the law of conservation of electric charge.

By adding (12.30) and (12.32) we get

$$\sum_{\ell, h} \sum (N_+ - N_- + N_{0+} - N_{0-}) = \text{constant} . \tag{12.34}$$

In particular, according as the quantity (12.34) is even or odd, we shall have

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[33] C.N.Yang and J.Tiomno, Phys.Rev. 79, 495(1950).

also

$$N = \sum_{\ell,h}\sum (N_+ + N_- + N_{0+} + N_{0-}) = \text{even or odd.} \tag{12.35}$$

This means that the parity of the total number of spinor particles conserves. This is the original superselection rule first introduced by W.W.W.[34] The generating unitary operator W can be obtained by putting $\beta = 0$, $\gamma = \delta = \pi$ in (12.26). Then (12.26) becomes Δ introduced in (2.82).

The W considered in (7.10) is obtained from (12.26) by putting $\beta = 0$ and equating $\gamma\ (= \delta)$ to $2(\alpha - \beta)$ of (7.10). The transformation considered in connection with (5.12) is essentially equivalent to the first line of (12.23). If we leave the value of γ in (5.16) or in (7.4) to be arbitrary, then the square of mirage operator becomes a W.

Since W(12.26) commutes with all the known physical quantities and the transition matrix U, we may consider the selection rules engendered by it are actually also superselection rules.

The arbitrariness of the phase-factor, which we have exploited in the foregoing, is not the only arbitrariness in the matrix representations of the emission and absorption operators. The absorption operator of the j-th spinor eigen-state, for instance, has the expression:

$$g_j = \prod_{i<j} (1 - 2N_i) \cdot f_j \cdot \prod_{k>j} 1\!\!1_k \tag{12.36}$$

with

$$(1 - 2N_i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i, \quad f_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1\!\!1_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_k. \tag{12.37}$$

We shall write, for brevity,

$$g_j = \Delta_j f_j, \qquad \Delta_j = \prod_{i<j} (1 - 2N_i). \tag{12.38}$$

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[34] G.C.Wick, A.S.Wightman and E.P.Wigner, Phys. Rev. 88, 101(1952).

If we extend i in (12.38) to all the eigen states, Δ_j becomes Δ of (2.82)

Now, it is obvious that this representation has a basic arbitrariness with regard to the ordering or labeling of the eigen-functions. In particular, it is of some interest to investigate the total inversion of the ordering. This amounts to putting $(1-2N_k)$ where we had unity matrices $\mathbb{1}_k$ in (12.36) and putting unity matrices $\mathbb{1}_i$ where we had $(1-2N_i)$ in (12.36). This can be done by simply multiplying (12.36) by Δ (2.82) from the left:

$$Y g_i Y^{-1} = \Delta g_i = -g_i \Delta \quad , \quad Y \bar{g}_i Y^{-1} = \bar{g}_i \Delta = -\Delta \bar{g}_i . \tag{12.39}$$

By a simple calculation, we can discover the explicit expression of Y.

$$Y = Y^{-1} = \prod_i^{\infty} (1 - N_i + \Delta_i N_i) \tag{12.40}$$

This unitary transformation as such does not, however, commute with all the physical quantities. If we get a unitary transformation W such that

$$W \psi W^{-1} = \Delta \psi \quad , \quad W \psi^{\times} W^{-1} = \psi^{\times} \Delta \tag{12.41}$$

this W will certainly commute with all the physical quantities. This is the type of transformation considered under (4.14). (12.41) will mean, in terms of "particles" and "antiparticles", that

$$\left.\begin{aligned} W g_{+i} W^{-1} &= \Delta g_{+i} \quad , & W g^{\times}_{+i} W^{-1} &= \bar{g}_{+i} \Delta \\ W g_{-i} W^{-1} &= g_{-i} \Delta \quad , & W g^{\times}_{-i} W^{-1} &= \Delta \bar{g}_{-i} \end{aligned}\right\} \tag{12.42}$$

This W can be given an explicit form:

$$W = W^{-1} = \prod_{i,j} (1 - N_{+i} + \Delta_{+i} N_{+i})(1 - N_{-j} - \Delta_{-j} N_{-j}) . \tag{12.43}$$

This W is again the generator of a superselection rule, but the physical conclusion that can be drawn from (12.43) brings nothing new. If we have a state

with N_+ "particles" and N_- "antiparticles", the eigen value of W(12.43) for such a state is

$$(-1)^{(N_+-N_-)(N_+-N_--1)/2} \tag{12.44}$$

This quantity certainly conserves if $(N_+ - N_-)$ conserves, (12.34).

#13. Illustrations -- Disintegration of Positronium and Neutral Pion

The purpose of this section is to show by examples how the general principles developed in the foregoing sections can be applied to concrete problems. It is therefore not intended to give any exhaustive study of the decay modes of the positronium and the neutral pion, but only to derive some of the simplest facts about these decay processes. Except for a few, results mentioned in this section have previously been discovered by other authors by more or less similar methods.[35]

Let us first determine the charge parity π_c , space parity π_s and x-parity π_x of positronium state Ψ_{pstr}:

$$C\,\Psi_{pstr} = \pi_c \Psi_{pstr} \quad ; \quad M\Psi_{pstr} = \pi_s \Psi_{pstr} \quad ; \quad M_x \Psi_{pstr} = \pi_x \Psi_{pstr}. \tag{13.1}$$

where C, M and M_x are given by (8.21), (8.26) and (8.34) using the angular momentum representation. The lowest positronium state Ψ_{pstr} may be a singlet state or a triplet state, i.e., we shall have an expansion of the type

$$\Psi_{pstr} = \sum a\, \bar{g}_+(\omega, \kappa, m, \rho)\, \bar{g}_-(\omega', \kappa', m', \rho')\, \Psi_{vac} \tag{13.2}$$

corresponding to the resultant total angular momentum zero or one.

However, for our purpose of determining the π's, we do not need to handle the general expression (13.2). From (8.21), (8.26) and (8.34), we see that C, M and M_x only change the charge and the magnetic quantum number and leave the other quantum numbers. Therefore, in an expression like (13.2), we have only to pick up, as representative terms, a small group of terms such that the effect of C, M and M_x is limited within this group. Since C and M must commute with the resultant angular momentum and its z-component of the compound system, the value of π_c and π_s determined with respect to a small group of representative terms must be the same as for the entire expression. The operation M_x is

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35. Among others, see C.N.Yang, Phys.Rev. 77, 242 (1950); L.Wolfenstein and D.G.Ravenhall, Phys.Rev.88, 279(1952); A.Pais and R.Yost, Phys.Rev.87, 871(1952).

commutable with the total angular momentum but not with its z-component, therefore π_x will have a meaning only for a state for which the z-component of the total angular momentum is zero.

The simplest representative terms in (13.2) would be those which correspond to a positron and a negatron, both in the same S-state: $\omega = \omega'$, $\varkappa = \varkappa' = -1$, $\rho = \rho' = 1$. Thus, the singlet state must include at least the following representative terms:

$$\Psi_{sing} = (1/\sqrt{2})\left[\bar{g}_+(\omega,-1,\tfrac{1}{2},1)\,\bar{g}_-(\omega,-1,-\tfrac{1}{2},1) - \bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\,\bar{g}_-(\omega,-1,\tfrac{1}{2},1)\right]\Psi_{vac} \qquad (13.3)$$

The triplet state will have the representative terms:

$$\Psi^{(1)}_{trip} = \bar{g}_+(\omega,-1,\tfrac{1}{2},1)\,g_-(\omega,-1,\tfrac{1}{2},1)\,\Psi_{vac} \qquad (13.4)$$

$$\Psi^{(0)}_{trip} = (1/\sqrt{2})\left[\bar{g}_+(\omega,-1,\tfrac{1}{2},1)\,\bar{g}_-(\omega,-1,-\tfrac{1}{2},1) + \bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\,\bar{g}_-(\omega,-1,\tfrac{1}{2},1)\right]\Psi_{vac} \qquad (13.5)$$

$$\Psi^{(-1)}_{trip} = \bar{g}_+(\omega,-1,-\tfrac{1}{2},1)\,\bar{g}_-(\omega,-1,-\tfrac{1}{2},1)\,\Psi_{vac} \qquad (13.6)$$

The superscript on Ψ_{trip} indicates the z-component of the resultant angular momentum.

By applying (8.21), (8.26) and (8.34) to (13.3), (13.4), (13.5) and (13.6), we can determine the π's in the sense of (13.1). The results are listed in Table VII.

	Ψ_{sing}	$\Psi^{(1)}_{trip}$	$\Psi^{(0)}_{trip}$	$\Psi^{(-1)}_{trip}$
π_c	+	−	−	−
π_s	−	−	−	−
π_x	−	×	+	×

Table VII. The charge parity π_c and the space parity π_s of the lowest positronium states.

$\Psi_{trip}^{(1)}$ and $\Psi_{trip}^{(-1)}$ are not eigen-states of M_x ; they are interchanged by application of M_x .

Remembering that the charge parity of a photon state is just the parity of the number of photons present, (6.32), we conclude that Ψ_{trip} cannot decay into a two-photon state. A two-photon decay then will suggest Ψ_{sing} for the positronium.

We shall next study two-photon states more closely using the representation used in Subsection #6.C. If we fix the coordinate system to the centre of mass of the decaying positronium, the two photons will have equal and opposite momenta, say k_z and $-k_z$. We can make four independent states.

$$\left.\begin{aligned} \Psi_1 &= (1/\sqrt{2})\left[\bar{g}(k_z,1)\,\bar{g}(-k_z,-1) + \bar{g}(k_z,-1)\,\bar{g}(-k_z,+1)\right]\Psi_{vac} \\ \Psi_2 &= (1/\sqrt{2})\left[\bar{g}(k_z,1)\,\bar{g}(-k_z,-1) - \bar{g}(k_z,-1)\,\bar{g}(-k_z,+1)\right]\Psi_{vac} \\ \Psi_3 &= \bar{g}(k_z,1)\,\bar{g}(-k_z,1)\Psi_{vac} \;; \qquad \Psi_4 = \bar{g}(k_z,-1)\,\bar{g}(-k_z,-1)\Psi_{vac} \end{aligned}\right\} \quad (13.7)$$

The states Ψ_3 and Ψ_4 do not interest us very much, because the z-component of the angular momentum is ± 2, which is certainly too large for comparison with the lowest positronium states or with the neutral pion at rest.

The charge parity π_c , space parity π_s and x-parity π_x are determined with the help of (6.36) and (6.38). The results are listed in Table VIII.

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
π_c	+	+	+	+
π_s	+	−	+	+
π_x	+	−	×	×
pol	//	⊥	×	×

Table VIII The charge parity π_c , space parity π_s and x-parity π_x of two-photon states. "pol" stands for planes of polarization of two photons.

Since π_s of positronium is -1, the only possible two-photon state into which it can decay is Ψ_2 . Since π_c of any two-photon state is $+1$, it can only originate from Ψ_{sing} . Thus, the two-photon decay of positronium has to be

$$\Psi_{sing} \rightarrow \Psi_2 \tag{13.8}$$

Before the fruitful application of charge-parity was introduced, they excluded $\Psi^{(0)}_{trip} \rightarrow \Psi_2$ by comparing π_x .

It is interesting to characterize Ψ_1 and Ψ_2 , (13.7), by the correlation of planes of linear polarization of the two photons involved. In (6.41), an operator $\Pi_x(k_z)$ was introduced, which measures the probability of a photon having momentum k_z and linear polarization in the x-direction. Then the operator:

$$\Pi_{\parallel} \equiv \Pi_x(k_z)\Pi_x(-k_z) + \Pi_y(k_z)\Pi_y(-k_z) \tag{13.9}$$

will measure the probability of two photons having momenta k_z and $-k_z$ and having parallel linear polarization. In the same way, for the probability of two photons being observed as having perpendicular linear polarization, we get an operator

$$\Pi_{\perp} \equiv \Pi_x(k_z)\Pi_y(-k_z) + \Pi_y(k_z)\Pi_x(-k_z) \tag{13.10}$$

By applying (13.9) and (13.10) to (13.7), we obtain with the help of (6.41),

$$\left.\begin{array}{ll} \Pi_{\parallel}\Psi_1 = \Psi_1 & \Pi_{\perp}\Psi_1 = 0 \\ \Pi_{\parallel}\Psi_2 = 0 & \Pi_{\perp}\Psi_2 = \Psi_2 \end{array}\right\} \tag{13.11}$$

Ψ_3 and Ψ_4 are not eigenfunctions of $\Pi_{\parallel}$ and $\Pi_{\perp}$, but we can make eigenfunctions by taking $\Psi_3+\Psi_4$ and $\Psi_3-\Psi_4$. This analysis shows that if we observe the correlation of linear polarization of two decay-photons of the positronium, they will turn out to be perpendicular to each other.[37]

It may be worth noting also that, in the angular momentum representation,

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[37] J. A. Wheeler, Ann. N.Y.Acad.Sci. 48, 219(1946).

a two-photon state can have odd parity only if the values of ℓ of the two photons are of different parity. See (6.24).

Let us next briefly review the decay of the neutral pion into photons. We have seen in #10 that the charge parity and the "kind" of the neutral pion are the same as their source, and they are given in Table IV. The "intrinsic" parity of the neutral pion is the space-parity of the state in which a single particle of this neutral field is present and at rest ($k = 0$). This sign is given by the symbol $(\pm)'$ in (5.23) or (5.32), and the double sign ($\pm$) in (5.16), and is determined by ρ_M in (10.10). The connection between the "kind" and the intrinsic parity is tabulated in the following table.

	scalar		vector	
kind	reg. 2nd	1st, 3rd	reg., 2nd	1st, 3rd
intrinsic parity	+	—	—	+

Table IX. Connection between the "kind" and the intrinsic parity.

This is nothing but a re-arrangement of a portion of Table III, Part I. In the case of a vector, the space-component determines the parity.

According to #10, there are actually six cases of $u^{(3)}$- field:

(1) scalar having (S)—interaction

(2) scalar having (V)—interaction

(3) vector having (V)- and (T)—interaction

(4) pseudo-scalar having (PS)—and (PV)—interaction

(5) pseudo-vector having (PV)—interaction

(6) pseudo-vector having (PT)—interaction

Mixing of (1) and (2), and mixing of (5) and (6) are prohibited on account of charge-symmetry and reversibility. Table X gives the intrinsic parity and charge parity of these six kinds of neutral pions, as can be derived from Tables IV. and IX .

	(1)	(2)	(3)	(4)	(5)	(6)
	S(S)	S(V)	V(V,T)	PS (PS,PV)	PV(PV)	PV(PT)
π_c	+	−	−	+	+	−
kind	reg.	2	2	1	1	3
π_s	+	+	−	−	+	+

Table X. The charge parity π_c and the intrinsic space-parity π_s of a neutral pion classified according to its source.

If experiment shows that the neutral pion decays into two photons, it means that $\pi_c = +1$, and the possible type of pion is limited to (1), (4) and (5). If further experiment shows that the planes of polarization of two photons are always perpendicular, it means $\pi_s = -1$, and the only possibility left is (4).

Appendix

It will be shown that by the use of 8-component spinors we can obtain a reversion operator such that the double-reversion becomes multiplication by a c-number, (3.45). Suppose we take, instead of (7.3),

$$\left.\begin{aligned} (R^{-1}\psi(t)R)^{T} &= e^{i\beta}E_0 K \Gamma \psi^{\times}(-t) \\ (R^{-1}\psi^{\times}(t)R)^{T} &= e^{-i\beta}\psi(-t)\Gamma^{-1}K^{-1}E_0 \end{aligned}\right\} \qquad \text{(A-1)}$$

where Γ is a unitary matrix of yet undetermined nature. This transformation will satisfy the requirements (7.1) if Γ commutes with the matrices with Dirac indices such as the E's and K and with a q-number bilinear in ψ and $\psi^{\times}$. If this is the case, the double-reversion will result in

$$R^{T}R^{-1}\psi R R^{-1T} = -\psi\Gamma^{-1}(R^{-1}\Gamma R)^{T}. \qquad \text{(A-2)}$$

If we can discover a Γ such that

$$(R^{-1}\Gamma R)^{T} = -\Gamma, \qquad \text{(A-3)}$$

then the double-reversion will become a c-number. We can think of two possibilities: (1) Γ is a matrix in the sense of a q-number in the quantized field theory, (2) Γ is a matrix with Dirac indices. In either case, a certain modification of the existing formalism may be necessitated. In this Appendix, we shall discuss the second alternative. In this case (A-3) becomes

$$\Gamma^{T} = -\Gamma \qquad \text{(A-4)}$$

i.e., Γ should be an antisymmatric matrix that commutes with the E-matrices.

It is well-known that if a matrix commutes with the four basic E's, then it must be just a number. However, this theorem is true only when the matrices have 4 columns and 4 rows. If we use 8-8-matrices, we can satisfy all the conditions imposed on Γ by putting, for instance,

$$E_\mu \rightarrow \begin{pmatrix} E_\mu & 0 \\ 0 & E_\mu \end{pmatrix}, \quad K \rightarrow \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad J \rightarrow \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \bar{\Gamma} = \Gamma^{-1} = \Gamma^{T} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{A.5}$$

By this doubling of the E-matrices, the number of eigenfunctions of the Hamiltonian for a given momentum $\mathbb{k}$ will become eight instead of the usual four. In other words, besides the charge (±) and spin (±), a new quantum number $\gamma = \pm 1$ intervenes corresponding to the hermitian operator $i\Gamma$:

$$i\Gamma\,\psi[\mathbb{k},\mu,\gamma=\pm 1] = \pm\psi. \tag{A.6}$$

We can for instance assume that $\gamma = 1$ represents the electron-positron states, while $\gamma = -1$ represents other fermion states, say, neutrino-antineutrino states. If we adopt this assignment, the current vector will become $i\psi^\times E_\mu (1+i\Gamma)/2\,\psi$ instead of $i\psi E_\mu \psi$. But (A.1) will still serve the purpose of reversion.

On account of (A.1), we should use $E_0 K\Gamma$ everywhere $E_0 K$ was used in the analysis of #7. By this substitution, however, (7.21) will still remain unchanged, for Γ commutes with all the operators O. The relation which should replace (7.24) will then read

$$E_0 K \Gamma\,\psi^\times[\mathbb{k},\mu,\gamma](x,t) = e^{i\alpha}\psi(-\mathbb{k},\mu,\gamma](x,-t) \tag{A.7}$$

If we solve this equation for $\psi[-\mathbb{k},\mu,\gamma]$ with the help of (2.20) which is still valid, we get

$$E_0 K \Gamma^{T} \psi^\times[-\mathbb{k},\mu,\gamma](x,-t) = -e^{i\alpha}\psi[\mathbb{k},\mu,\gamma](x,-t). \tag{A.8}$$

In contrast to the case of (7.24) and (7.25), (A.7) and (A.8) are completely symmetrical for the interchange of $\mathbb{k}$ with $-\mathbb{k}$ and t with -t. As a consequence, we do not need to introduce τ (7.26). Instead of (7.29), we shall

have

$$\psi[\mathbb{k},\mu,\gamma] = \alpha[\mathbb{k},\mu,\gamma] \exp(+-) \tag{A.9}$$

and instead of (7.30),

$$\begin{aligned}\psi(x,t) = V^{-\frac{1}{2}} \sum_{\mathbb{k}}\sum_{\tau}\sum_{\gamma} \big[& g_+(\mathbb{k},\mu,\gamma)\,\alpha[\mathbb{k},\mu,\gamma]\exp(+-) \\ & + \bar{g}_-(\mathbb{k},\mu,\gamma)\,\alpha^{\times}[\mathbb{k},\mu,\gamma]\exp(-+)\big] .\end{aligned} \tag{A.10}$$

Applying (A.1) on (A.10), we obtain with the help of (A.7) and (A.9), with $\alpha = 0$,

$$\left(R^{-1} g_\pm(\mathbb{k},\mu,\gamma) R\right)^T = e^{\pm i\beta}\, \bar{g}_\pm(-\mathbb{k},\mu,\gamma) , \tag{A.11}$$

which should replace (7.37). The explicit expression of R to take the place of (7.38) (7.40) will then be

$$R = R_0 R' \quad ; \qquad R_0 = \prod_{\pm}\prod_{\mathbb{k}}\prod_{\mu}\prod_{\gamma} e^{\pm i\beta N_\pm(\mathbb{k},\mu,\gamma)} \tag{A.12}$$

$$\left.\begin{aligned} R' = \prod_{\pm}\prod_{\mathbb{k}}{}'\prod_{\mu}\prod_{\gamma} \big[& 1 - N_\pm(\mathbb{k},\mu,\gamma) - N_\pm(-\mathbb{k},\mu,\gamma) \\ & + \bar{g}_\pm(\mathbb{k},\mu,\gamma)\, g_\pm(-\mathbb{k},\mu,\gamma) + \bar{g}_\pm(-\mathbb{k},\mu,\gamma)\, g_\pm(\mathbb{k},\mu,\gamma)\big] \end{aligned}\right\} \tag{A.13}$$

This R certainly satisfies the expected condition:

$$R^T = R . \tag{A.14}$$

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